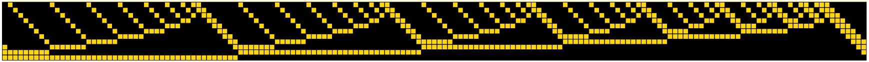

An explicit universal cycle for the $(n-1)$-permutations of an n-set

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An example with $n = 3$

- Consider the circular string

$$321312$$

- Its length 2 substrings are

$$32, 21, 13, 31, 12, 23.$$

  These are the 2-permutations of a 3-set.

- In general, we want a circular string of length $n!$ such that every k-permutation of $[n] = \{1, 2, \ldots, n\}$ occurs (uniquely) as a substring.
- Such strings were shown to exist by Brad Jackson, *Universal cycles of k-subsets and k-permutations*, Discrete Mathematics, 149 (1996) 123–129.

Knuth's challenge

The problem for $k = n - 1$ is discussed by D.E. Knuth, *The Art of Computer Programming, Volume 4, Generating All Tuples and Permutations*, Fascicle 2, in Exercise 112 of Section 7.2.1.2. On page 121 we find the following quote:

> *"At least one of these cycles must almost surely be easy to describe and to compute, as we did for de Bruijn cycles in Section 7.2.1.1. But no simple construction has yet been found."*

We present here a simple (and elegant and efficient) construction.

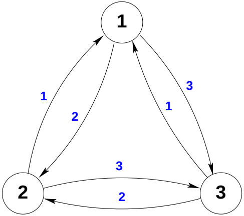
The underlying graph

- The Jackson graph $J_{k,n}$: vertices are the $(k-1)$-permutations of $[n] = \{1, 2, \ldots, n\}$ and directed edges are of the form

$$a_1 a_2 \cdots a_{k-1} \longrightarrow a_2 \cdots a_{k-1} b$$

  for $b \in [n] \setminus \{a_2, \cdots, a_{k-1}\}$.
- Each vertex has in-degree and out-degree $n - k + 1$.
- The graph is vertex-transitive.
- The graph is Eulerian. To prove it you need to show that it is strongly-connected.

Example: $J_{2,3}$.



One Eulerian cycle is our initial example 321312 (starting at vertex 2).

$n \cdot (n-1) \cdots 3 \cdot 2 = n!$

- By adding the missing numbers, a $(n-1)$-permutation of $[n]$ becomes a permutation of $[n]$.
- $$\begin{array}{ccc} 32 & \to & 321 \\ 21 & \to & 213 \\ 13 & \to & 132 \\ 31 & \to & 312 \\ 12 & \to & 123 \\ 23 & \to & 231 \end{array}$$
- $$3715264 \nearrow 7152634 \qquad 3715264 \searrow 7152643$$

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- In the universal cycle U_7:

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- In the universal cycle U_7:

$$\begin{array}{ccc} & \nearrow & 715263\mathbf{4} \\ 371526\mathbf{4} & & \\ & \searrow & 715264\mathbf{3} \end{array}$$

$n \cdot (n-1) \cdots 3 \cdot 2 = n!$

- By adding the missing numbers, a $(n-1)$-permutation of $[n]$ becomes a permutation of $[n]$.
- $$\begin{array}{ccc} 32 & \longrightarrow & 321 \\ 21 & \longrightarrow & 213 \\ 13 & \longrightarrow & 132 \\ 31 & \longrightarrow & 312 \\ 12 & \longrightarrow & 123 \\ 23 & \longrightarrow & 231 \end{array}$$
- As permutations:

$$\begin{array}{ccc} & \nearrow & 715263\textcolor{green}{4} \\ 371526\textcolor{green}{4} & & \\ & \searrow & 715264\textcolor{green}{3} \end{array}$$

The Cayley Graph Connection

- 3715264 $\nearrow$ 7152634 $\quad \sigma_6 = \sigma_{n-1}$

  $\searrow$ 7152643 $\quad \sigma_7 = \sigma_n$

- Define

$$\Xi_n := \overrightarrow{\mathrm{Cay}}(\{\sigma_n, \sigma_{n-1}\} : \mathbb{S}_n)$$

- The problem of finding a Hamilton cycle in Ξ_n is equivalent to finding a universal cycle of $(n-1)$-permutations of an n-set.
- (Which is equivalent to finding an Eulerian cycle in $J_{k,n}$.)

Our construction, as a bitstring

Consider the binary string S_n defined by the following recursive rules. The base case is $S_2 = 00$. Let $S_n = x_1 x_2 \cdots x_{n!}$ where $\overline{x}$ denotes flipping the bit x. Then, for $n > 2$,

$$S_{n+1} := \underbrace{001^{n-2}\ \overline{x}_1}\,\underbrace{001^{n-2}\ \overline{x}_2}\cdots\underbrace{001^{n-2}\ \overline{x}_{n!}}\,.$$

Examples:

$$\begin{aligned}
S_3 =&\quad 00\ \bar{0}\ 00\ \bar{0} \quad = \quad 00\ 1\ 00\ 1. \\
S_4 =&\quad 001\ \bar{0}\ 001\ \bar{0}\ 001\ \bar{1}\ 001\ \bar{0}\ 001\ \bar{0}\ 001\ \bar{1} \\
=&\quad 001\ 1\ 001\ 1\ 001\ 0\ 001\ 1\ 001\ 1\ 001\ 0. \\
S_5 =&\ (\ 0011\ 1\ 0011\ 1\ 0011\ 0\ 0011\ 0\ 0011\ 1\ 0011\ 1 \\
&\quad 0011\ 0\ 0011\ 0\ 0011\ 1\ 0011\ 1\ 0011\ 0\ 0011\ 1\ )^2
\end{aligned}$$

As a "morphism": $0 \mapsto 001^{n-2}1$ and $1 \mapsto 001^{n-2}0$

Our construction, as sequence of generators

Now define the mapping ϕ by $0 \to \sigma_n$ and $1 \to \sigma_{n-1}$ where $\sigma_k = (k \ \cdots \ 2\ 1)$.

Theorem

The list $\phi(S_n)$ is a Hamilton cycle in the directed Cayley graph Ξ_n.

Proof: Our proof is by induction on n. We construct a list $\Pi(n)$ of the permutations along such a Hamilton cycle. Construction:

$$\Pi(n)_{jn} := n\Pi(n-1)_j = n\pi.$$

$$\underbrace{\sigma_n(n\pi),\ \sigma_n^2(n\pi)}_{2},\ \underbrace{\sigma_{n-1}(\sigma_n^2(n\pi)),\ \ldots,\ \sigma_{n-1}^{n-3}(\sigma_n^2(n\pi))}_{n-3}.$$

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Starting with $\Pi_3 = 321, 213, 132, 312, 123, 231$:

4321	σ_4	4312	σ_4
3214	σ_4	3124	σ_4
2143	σ_3	1243	σ_3
1423	σ_3 ??	2413	σ_3 ??
4213	σ_4	4123	σ_4
2134	σ_4	1234	σ_4
1342	σ_3	2341	σ_3
3412	σ_3 ??	3421	σ_3 ??
4132	σ_4	4231	σ_4
1324	σ_4	2314	σ_4
3241	σ_3	3142	σ_3
2431	σ_4 ??	1432	σ_4 ??

The next list is 4321, 3214, 2143, etc.

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1243 σ_3
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1234 σ_4
2341 σ_3
3421 σ_3 ??

4231 σ_4
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- We noted before that every permutation appears on the list exactly once.
- But what is happening at the interfaces? Suppose $\pi = \Pi(n-1)_j = a\tau z$ and $\pi' = \Pi(n-1)_{j+1}$. Inductively, either
  Case A: $\pi' = \sigma_{n-1}(\pi) = \tau za$, or
  Case B: $\pi' = \sigma_{n-2}(\pi) = \tau az$.
- Last permutation on list for π.

$$\begin{aligned} \sigma_{n-1}^{n-3}(\sigma_n^2(na\tau z)) &= \sigma_{n-1}^{-2}(\sigma_n^2(na\tau z)) \\ &= \sigma_{n-1}^{-2}(\tau zna) \\ &= zn\tau a. \end{aligned}$$

- **Case A:** $\sigma_{n-1}(zn\tau a) = n\tau za = n\sigma_{n-1}(\pi) = n\pi'$, or
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000	3	000	. . 0	4321	4	0
001	3	001	. . 0	3214	3	1
002	3	002	. . 1	2143	2	2
003	2	003	. 1 .	1423	1	3
010	3	013	. . 0	4213	4	4
011	3	012	. . 0	2134	2	5
012	3	011	. . 1	1342	1	6
013	2	010	. 1 .	3412	3	7
020	3	020	. . 0	4132	4	8
021	3	021	. . 0	1324	1	9
022	3	022	. . 1	3241	3	10
023	1	023	0 . .	2431	2	11
100	3	123	. . 0	4312	4	12
101	3	122	. . 0	3124	3	13
102	3	121	. . 1	1243	1	14
103	2	120	. 1 .	2413	2	15
110	3	110	. . 0	4123	4	16
111	3	111	. . 0	1234	1	17
112	3	112	. . 1	2341	2	18

- Counting in base $2 \times 3 \times 4$.
- Position of change, R_4.
- Gray code in base $2 \times 3 \times 4$.
- The list S_4.
- $\mathbb{S}_4$
- U_4
- Rank

The counting algorithm

Multi-radix, with $0 \leq a_j \leq n - j$.

$a_{n+1} a_n \cdots a_1 \leftarrow 0\ 0\ \cdots\ 0$;
repeat
  $j \leftarrow 1$;
  while $a_j = n - j$ **do** $a_j \leftarrow 0$; $j \leftarrow j + 1$; **od**;
  output($[\![j \text{ even } \oplus a_j \leq 1]\!]$);
  $a_j \leftarrow a_j + 1$;
until $j \geq n$;

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The loopless version

Multi-radix, with $0 \leq a_j \leq n-j$.

$$a_{n+1}a_n \cdots a_1 \leftarrow 0\ 0\ 0\ \cdots\ 0;$$
$$d_n d_{n-1} \cdots d_1 \leftarrow 1\ 1\ 1\ \cdots\ 1;$$
$$f_n f_{n-1} \cdots f_1 \leftarrow n{+}1\ n{-}1\ n{-}2\ \cdots\ 1;$$

repeat

$j \leftarrow f_1;\ f_1 \leftarrow 1;$

output($[\![\, j \text{ even} \oplus (a_j - d_j \leq 0 \ \textbf{or}\ a_j - d_j \geq n-j) \,]\!]$);

$a_j \leftarrow a_j + d_j;$

if $a_j = 0$ **or** $a_j = n-j$

then $d_j \leftarrow -d_j;\ f_j \leftarrow f_{j+1};\ f_{j+1} \leftarrow j+1;$ **fi**;

until $j \geq n$;

The loopless version

Multi-radix, with $0 \le a_j \le n-j$.

$a_{n+1} a_n \cdots a_1 \leftarrow 0\ 0\ 0\ \cdots\ 0;$
$d_n d_{n-1} \cdots d_1 \leftarrow 1\ 1\ 1\ \cdots\ 1;$
$f_n f_{n-1} \cdots f_1 \leftarrow n{+}1\ \ n{-}1\ \ n{-}2\ \cdots\ 1;$
repeat
  $j \leftarrow f_1;\ f_1 \leftarrow 1;$
  output($[\![\ j$ even $\oplus (a_j - d_j \le 0$ **or** $a_j - d_j \ge n-j)\ ]\!]$);
  $a_j \leftarrow a_j + d_j;$
  if $a_j = 0$ **or** $a_j = n - j$
    then $d_j \leftarrow -d_j;\ f_j \leftarrow f_{j+1};\ f_{j+1} \leftarrow j+1;$ **fi**;
until $j \ge n$;

How many σ_n's are used?

The number, call it f_n, of σ_n's in $\phi(S_n)$ satisfies the recurrence relation

$$f_{n+1} = \begin{cases} 2 & \text{if } n = 1 \\ 3n! - f_n & \text{if } n > 1. \end{cases}$$

Iterate:

$$f_n = 2(-1)^n - 3\sum_{k=1}^{n-1}(-1)^k(n-k)!,$$

Thus:

$$f_n \sim 3(n-1)! \quad \text{or} \quad \frac{f_n}{n!} \sim \frac{3}{n}.$$

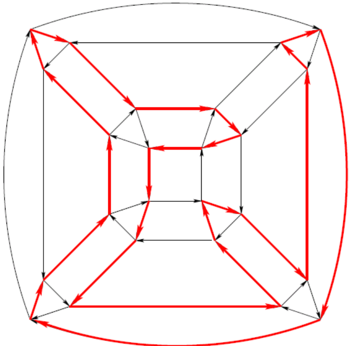
Appears in OEIS as A122972$(n+1)$ as the solution to the "symmetric" recurrence relation
$a(n+1) = (n-1)\cdot a(n) + n\cdot a(n-1)$.
The values of f_n for $n = 1..10$ are 1, 2, 4, 14, 58, 302, 1858, 13262, 107698, 980942.

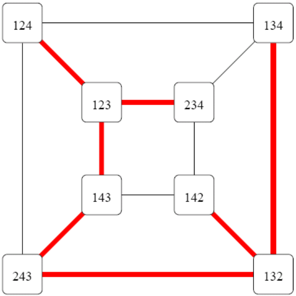
A lower bound on the number of σ_n's

- Observe: In any Hamilton cycle $|\sigma_{n-1}| \geq (n-1)!$ and $|\sigma_n| \geq n(n-2)!$.
- Improvement: In any Hamilton cycle $|\sigma_n| \geq 2n(n-2)! - 2$. Note that

$$\sigma_{n-1}^{-}\sigma_n\sigma_{n-1}^{-}\sigma_n = (n{-}1\ n)(n{-}1\ n) = id.$$

Contract the $n(n-2)!$ cosets of σ_{n-1}. Hamilton cycle in Ξ_n becomes a connected spanning subgraph.





Ranking

$$\operatorname{rank}(a_1 a_2 \cdots a_{k-1} n a_{k+1} \cdots a_n)$$

$$= \begin{cases} 0 & \text{if } n = 1, \\ n \cdot \operatorname{rank}(a_2 a_3 \cdots a_n) & \text{if } k = 1, \\ n - k + 1 + n \cdot \operatorname{rank}(a_n a_{k+1} \cdots a_{n-1} a_1 \cdots a_k) & \text{if } k > 1. \end{cases}$$

The expression $n - k + 1$ accounts for the position of the n, and the rest comes from the recursive part of the definition of $\Pi(n)$.

$$\operatorname{rank}(\ \alpha n \beta\ ) = \begin{cases} 0 & \text{if } \alpha = \beta = \epsilon, \\ n \cdot \operatorname{rank}(\ \beta\ ) & \text{if } \alpha = \epsilon, \\ n - |\alpha| + n \cdot \operatorname{rank}(\ \sigma(\beta)\alpha\ ) & \text{otherwise}\ , \end{cases} \qquad (1)$$

where $\sigma(\beta)$ is β rotated one position to the right.

Open problems

- Can the results of this paper be extended in some natural way to k-permutations of $[n]$ for $3 \leq k < n-1$?
- Among all Hamilton cycles in Ξ_n we determined the least number of σ_n edges that need to be used in a Hamilton cycle in Ξ_n. What is the least number of σ_{n-1} edges that need be used? In our construction, the number of σ_n edges is asymptotic to $3/n$ and the number of σ_{n-1} edges is asymptotic to $(n-3)/n$. Is there a general construction that uses more σ_n edges than σ_{n-1} edges?
- It would be interesting to gain more insight in to the ranking process. Is there a way to iterate the recursion so that it can be expressed as a sum?

Open problems, continued

- Can the results of this paper be extended to the permutations of a multiset? That is, given multiplicities $n_0, n_1, \ldots, n_t$, where n_i is the number of times i occurs in the multiset and $n = n_0 + n_1 + \cdots + n_t$, is there a circular string $a_1 a_2 \cdots a_N$ of length $N = \binom{N}{n_0, n_1, \ldots, n_t}$ with the property that

$$\{a_i \; a_{i+1} \; \cdots \; a_{i+n-2} \; \iota(a_i, a_{i+1}, \ldots, a_{i+n-2}) \mid 1 \leq i \leq N\}$$

  is equal to the set of all permutations of the multiset. Since the length of $a_i a_{i+1} \cdots a_{i+n-2}$ is $n-1$ it is not a permutation of the multiset; one character is missing. The function ι gives the missing character. We call these strings *shorthand universal cycles*. The current paper gave a shorthand cycle for permutations of $[n]$.

News flash

Dear Frank,

I finally have gotten Section 7.1.4 to the point where I could take a small breath and look at the mail that has come in since last summer about the other fascicles and prefascicles.

One of the most exciting things, of course, was to learn about Aaron's nice explicit universal cycles of permutations. In the next printing of Volume 4 Fascicle 2 I shall replace exercise 7.2.1.2–112 by two exercises, 112 and 113; 112 asks for (and gives hints towards) Aaron's explicit construction, while 113 is the former 112. These updates will be posted in the TAOCP errata listing all4f2.ps, later this week. I also stuck in a very brief mention of the multiset case, although you have apparently not yet written that paper.

Beautiful: stringology is really coming of age!

...

Thanks again for keeping me informed.

Best regards, Don

The end

Thanks for coming!