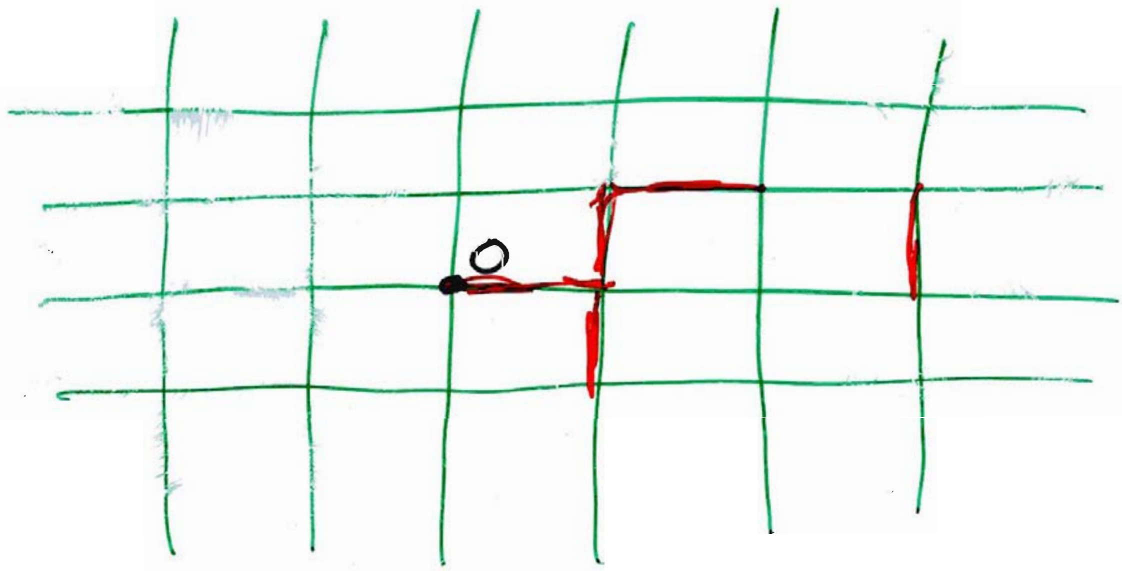
Percolation

Basic problem: each edge of infinite grid is open with probability p (independently).

What is probability $P_n(p)$ that the open cluster through the origin has size at least n?



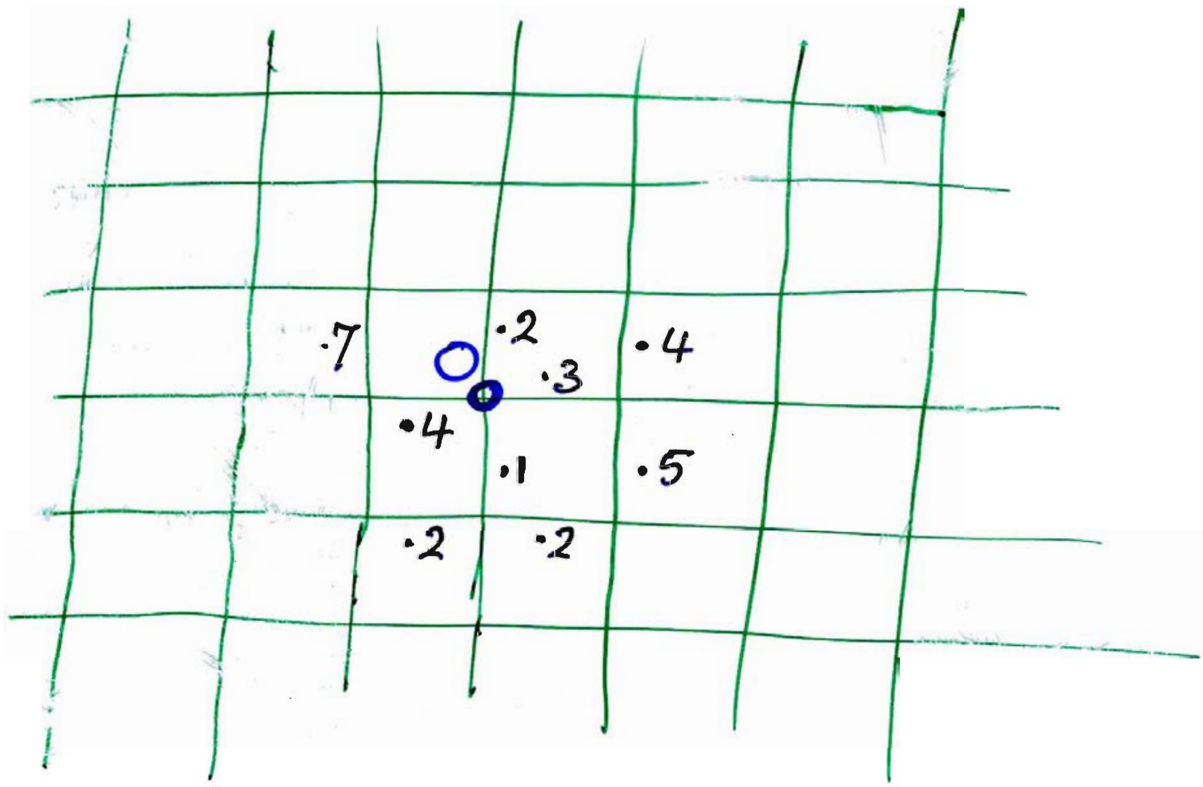
$$P_2(p) = 1 - (1-p)^4$$

$$P_n(p) \to P(p)$$

and $P(p)$ is the percolation probability

Hammersley idea

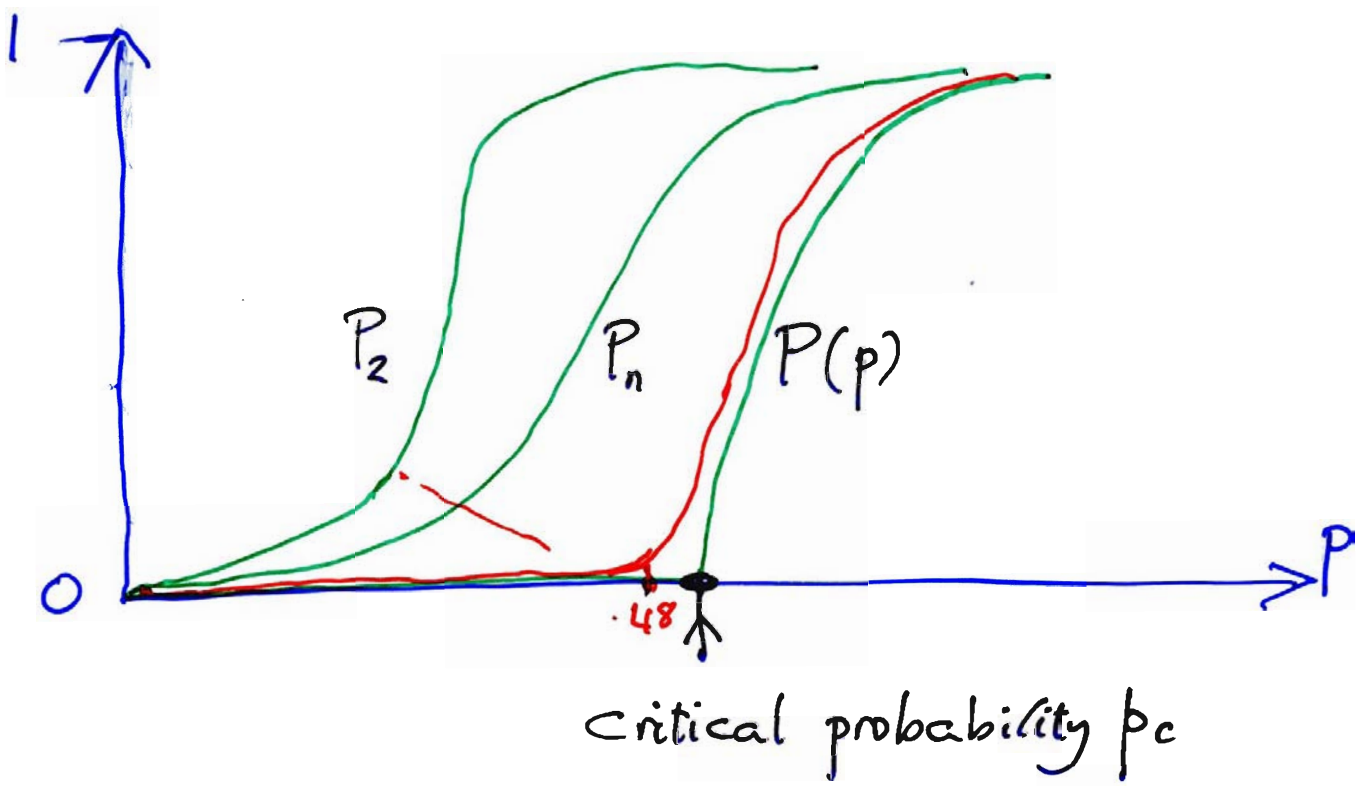
Give each edge e a "random" length z_e chosen uar from $(0,1)$



Interpret $z_e = \cdot 2$ as

"only fluid with value $\leq \cdot 2$" can traverse that edge.

Hence get "histogram" from one set of assigned z_e.



? Monte carlo method to estimate p_c and perhaps $P(p)$

Naive - brute force
- lots of samples

Finite Markov Chains

Let Z_n denote Markov chain on finite state space Ω with transition probabilites

$$p_{ij} = \Pr\{Z_{n+1} = j \mid Z_n = i\}.$$

$\underset{\sim}{P} = (p_{ij})$ is the transition matrix

and

$$p_{ij}(n) = \Pr\{Z_{a+n} = j \mid Z_a = i\}$$
$$= (\underline{P}^n)_{ij}.$$

Chain is irreducible if for each i, j $p_{ij}(n) > 0$ for some $n = n(i,j)$.

Aperiodic if for some n some j

$$\Pr(X_n = j \mid X_0 = j) > 0$$

and $\Pr(X_{n+1} = j \mid X_0 = j) > 0$.

Finite, irreducible and aperiodic

$\Rightarrow$ chain is <u>ergodic</u>.

$$\Rightarrow \lim_{n\to\infty} \Pr(X_n = j) = \pi_j > 0$$

and <u>stationary probabilities</u> $\{\pi_j\}$

are given by unique solution to

$$\pi_j = \sum_i \pi_i p_{ij}$$

$$\sum \pi_i = 1 .$$

<u>Note</u> : If transition matrix P is <u>symmetric</u> then stationary dist^n is <u>uniform</u>, that is

$$\pi_i = \pi_j \quad \forall\, i, j \in \Omega .$$

Simple Markov Chain Algorithm

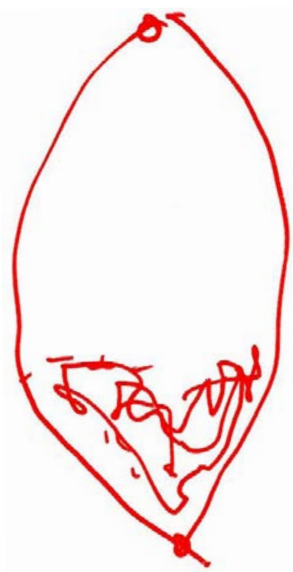
Typical question:

1) What does random planar graph on n vertices look like?

2) . . . . random triangle free graph on n vertices look like?

Basic Idea: Do simple random walk on all n-vertex planar graphs.

Hope it converges quickly to its stationary distribution.



How to Generate ?

Markov chain algorithm

(Denise Vasconcellos W 96)

(1) Start with G_0 any planar graph on vertex set $\{1, \dots, n\}$

(2) Pick distinct (a, b) uniformly at random from $\{1, \dots, n\}$

(3) If (a, b) is edge of G_t

then $G_{t+1} = G_t \setminus (a, b)$

(4) If (a, b) is not an edge of G_t and $G_t + (a, b)$ is still planar

$$G_{t+1} = G_t + (a, b)$$

Else

$$G_{t+1} = G_t$$

Properties of Markov Chain

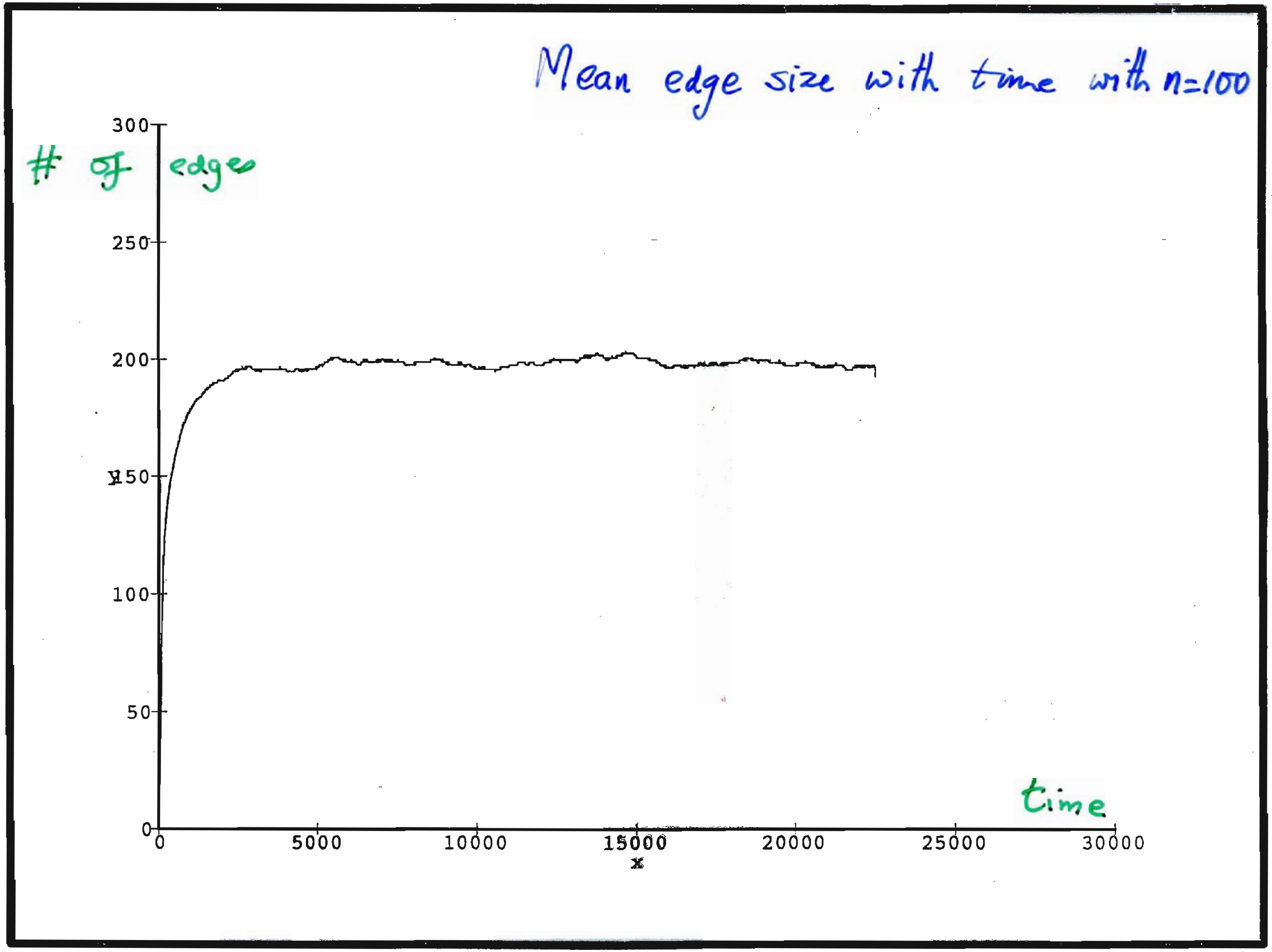
(1) States are all labelled planar graphs on n vertices

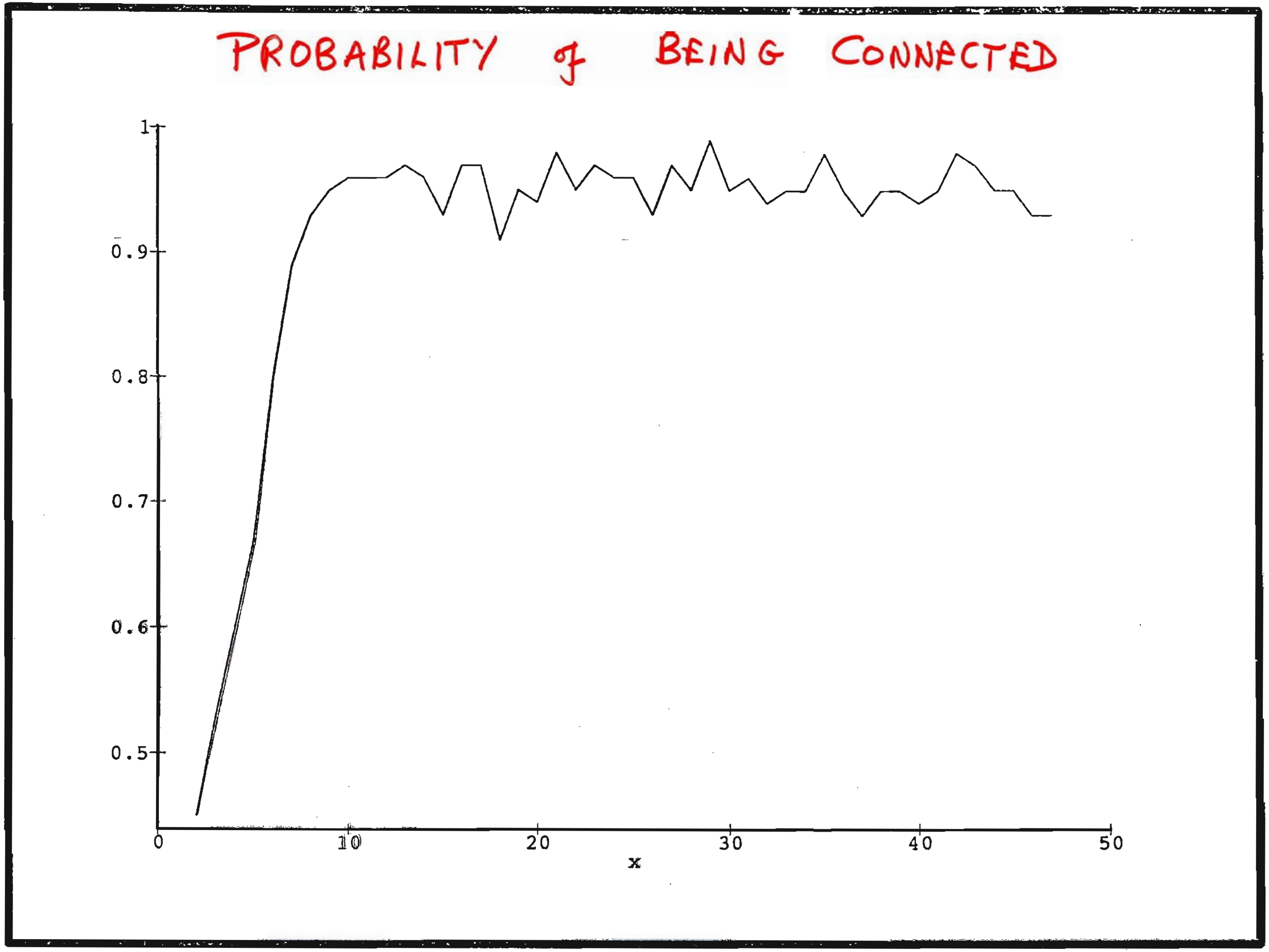
So if $n = 10$ there are

$$32099977749284$$

states.

(2) Transition matrix is irreducible symmetric so it has a UNIQUE stationary distribution which is UNIFORM over all states.


Mean edge size with time with n=100
# of edges
300
250
200
y150
100
50
0
0
5000
10000
15000
20000
25000
30000
x
time


PROBABILITY of BEING CONNECTED
1
0.9
0.8
0.7
0.6
0.5
0
10
20
30
40
50
x

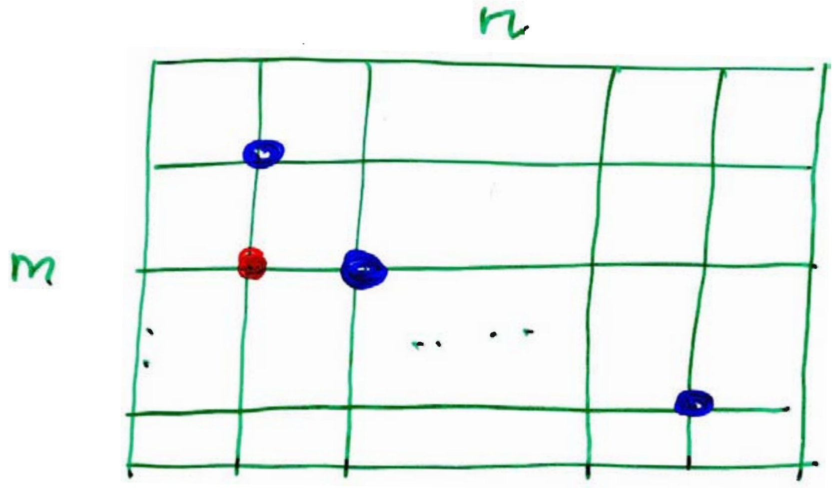
Performance of Algorithm?

(1) No known proof that this Markov chain converges rapidly to the limiting uniform distribution.

(2) It generalises in the obvious way to other hereditary properties but be careful.

(3) After a finite time it will only give an approximately uniform distribution

Voter Problem – Toroidal Grid



Initially k black vertices
$N-k$ white vertices.

At each instant of time
pick vertex v uar
choose neighbour u uar
change colour of v to that of u.

Th^m 1 With probability 1 end up with all black or all white.

Th^m 2 Probability consensus is all black $= k/N$
regardless of initial configuration.

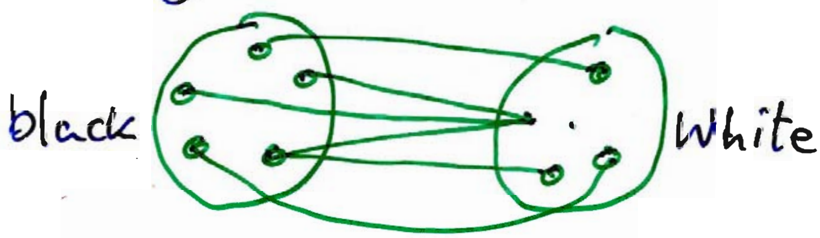
Antivoter Model on general graph G

(Donnelly + W)

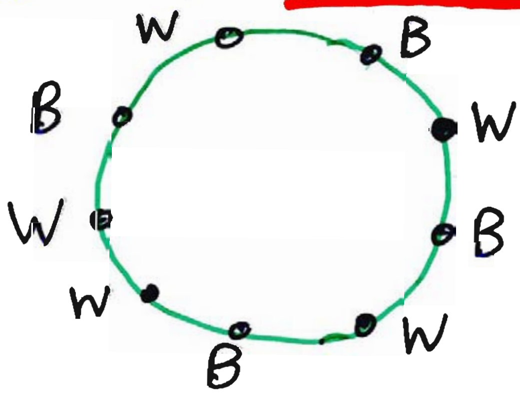
Same model except at each instant chosen vertex v chooses opposite colour to that of neighbour u.

Three scenarios

1) Graph G is bipartite – end with a "good" 2-colouring



2) G is an odd circuit



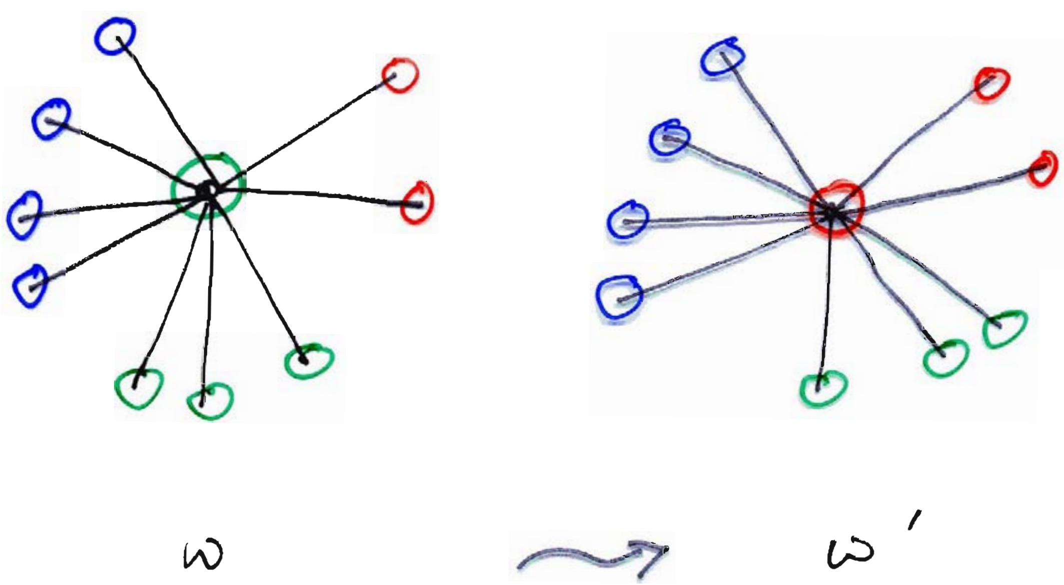
3) For general G end up in unknown stationary distribution

A 3-colouring algorithm (Petford-Welsh 85)

Ω = all possible 3-colourings of G

Given colouring ω pick vertex v and colour c from $\{R, B, G\}$ at random.

If $c = R$ change colour of v to R with probability $\propto \theta^{|\partial_R(v)|}$ where $0 < \theta < 1$



$$\Pr\{\omega \rightsquigarrow \omega'\} = \frac{1}{3n} \frac{\theta^2}{\theta^4 + \theta^2 + \theta^3}$$

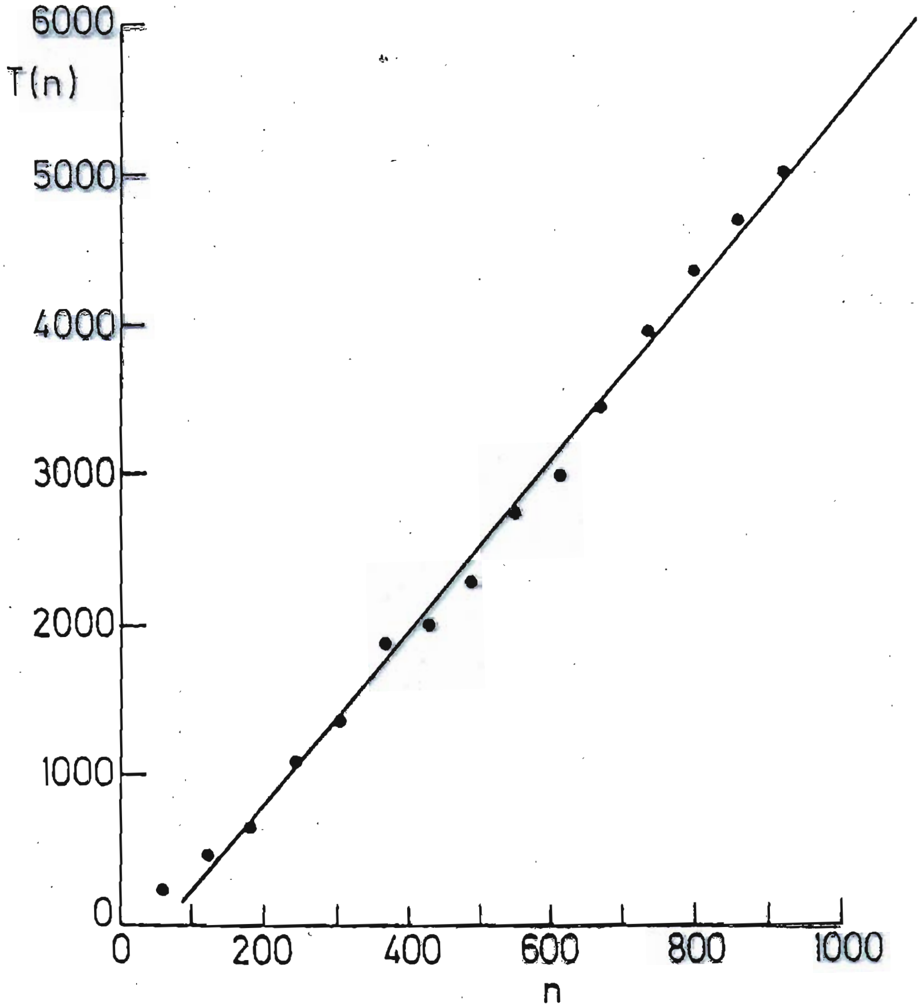


Fig. 2. $p = 0.5$

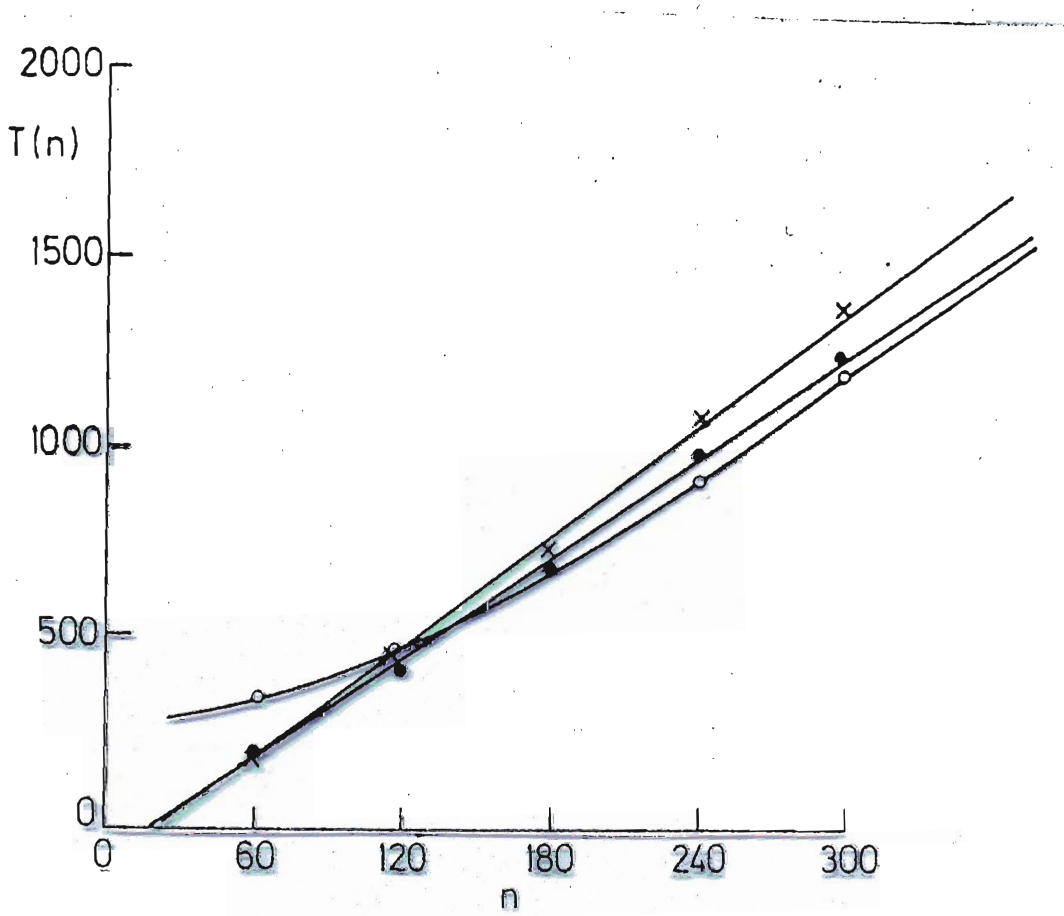


Fig. 1. Data from 100 trials for each value of n.

● $p = 0.5$

× $p = 0.9$

○ $p = 0.3$

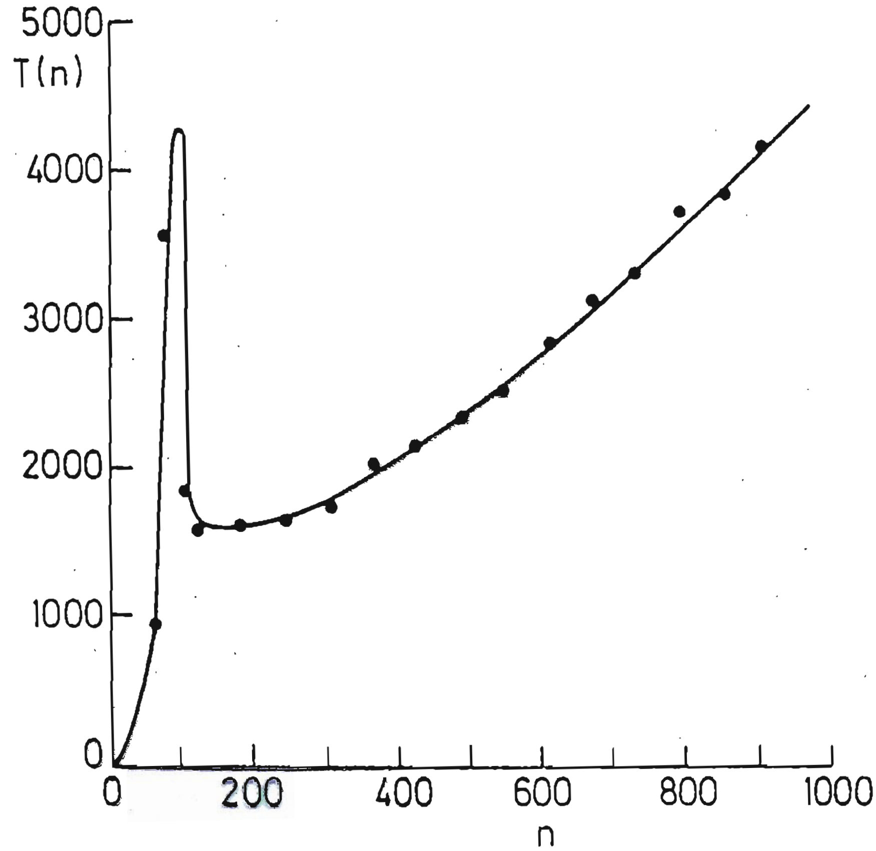


Fig. 3. $p = 0.1$

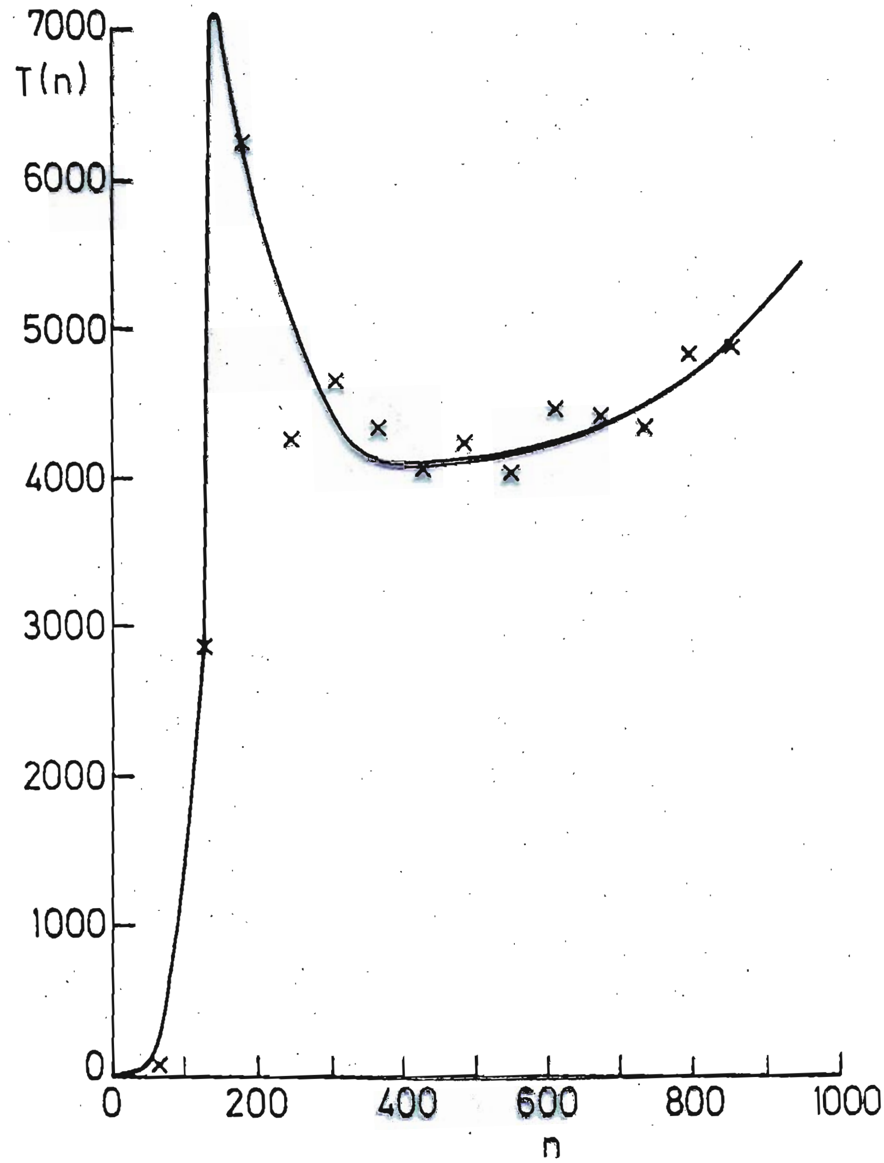


Fig. 5. $p = 0.05$

Time Reversibility

Suppose we can find positive a_i such that transition probabilite p_{ij} satisfy

$$a_i p_{ij} = a_j p_{ji} \qquad i \neq j$$

and

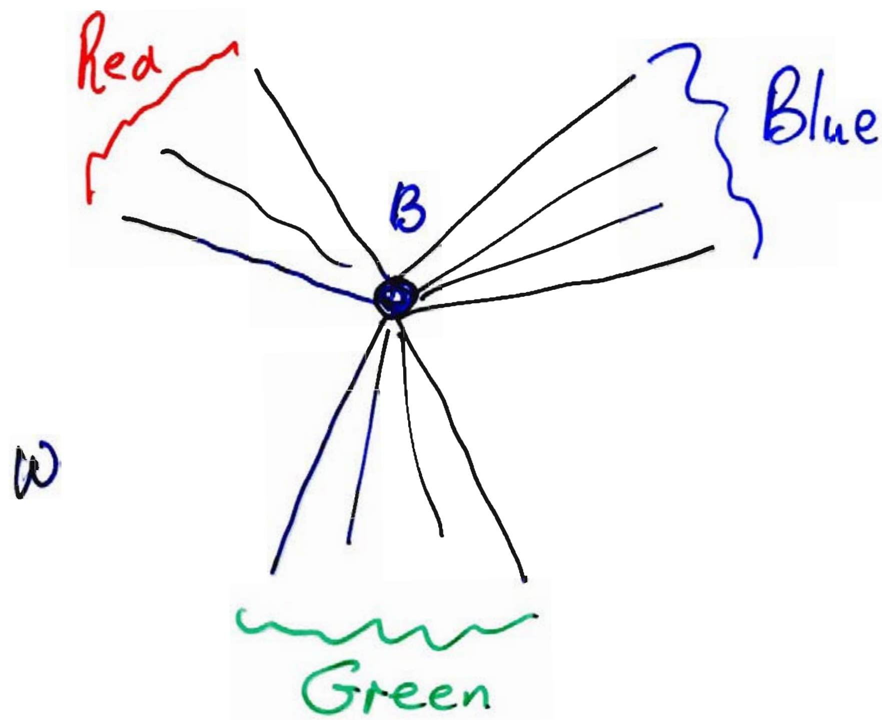
$$\sum a_i = 1.$$

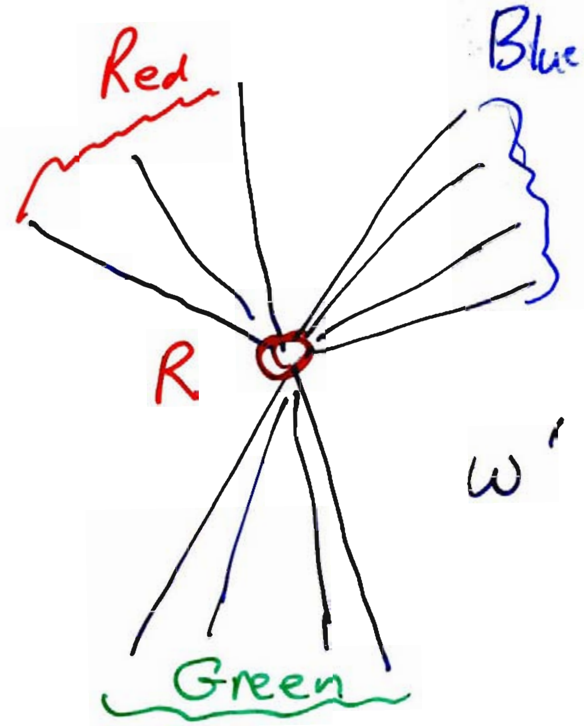
Then summing over i gives

$$\sum_i a_i p_{ij} = a_j \sum_i p_{ji}$$

$$= a_j$$

Hence $\{a_i\}$ must be the unique stationary distribution $\{\pi_j\}$.





Consider $\pi(\omega) \propto \theta^{\text{bad}(\omega)}$

$\text{bad}(\omega)$ = number of bad edges in colouring ω

$$\Pr\{\omega \rightsquigarrow \omega'\} \propto \theta^{|\partial_R|}$$

$$\therefore \pi(\omega)\, P(\omega \rightsquigarrow \omega') \propto \theta^{\text{bad}(\omega) + |\partial_R|}$$

$$= \theta^{\text{bad}(G\setminus v) + |\partial_B| + |\partial_R|}$$

$$= \pi(\omega')\, P(\omega' \rightsquigarrow \omega)$$

Hence $\pi(\omega) \propto \theta^{\text{bad}(\omega)}$ is

the stationary distribution

Consequence

"For $\Theta < 1$ Markov chain will tend to colourings with few bad edges"

Experimentally $\Theta \sim 1/4$ seemed best for 3-colouring problem.

Note Zerovnik has implemented algorithm for general $k > 3$ and qualitatively similar behaviour.

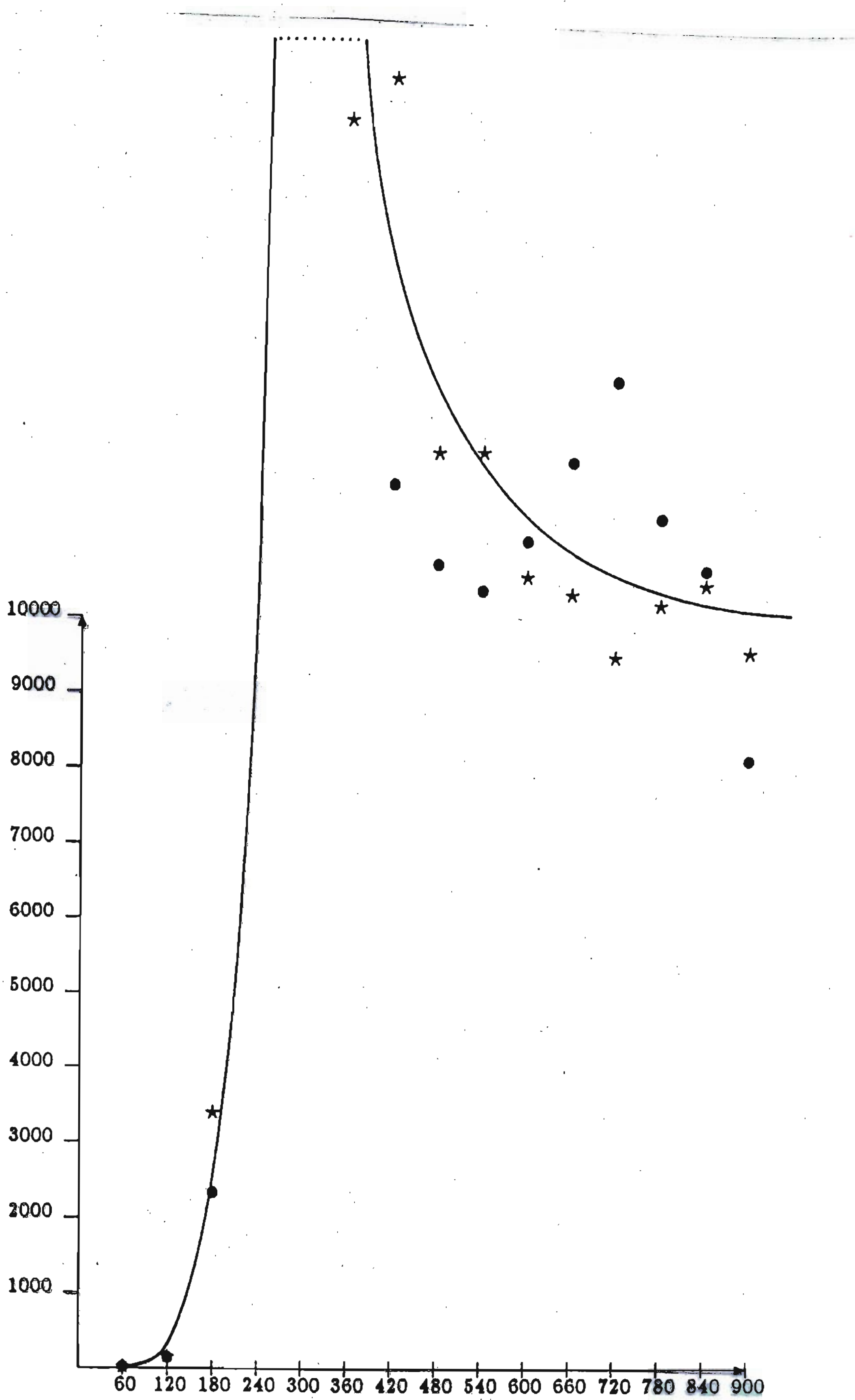


Fig. 4. $p=0.050$, $k=4$ (* – second set of samples).

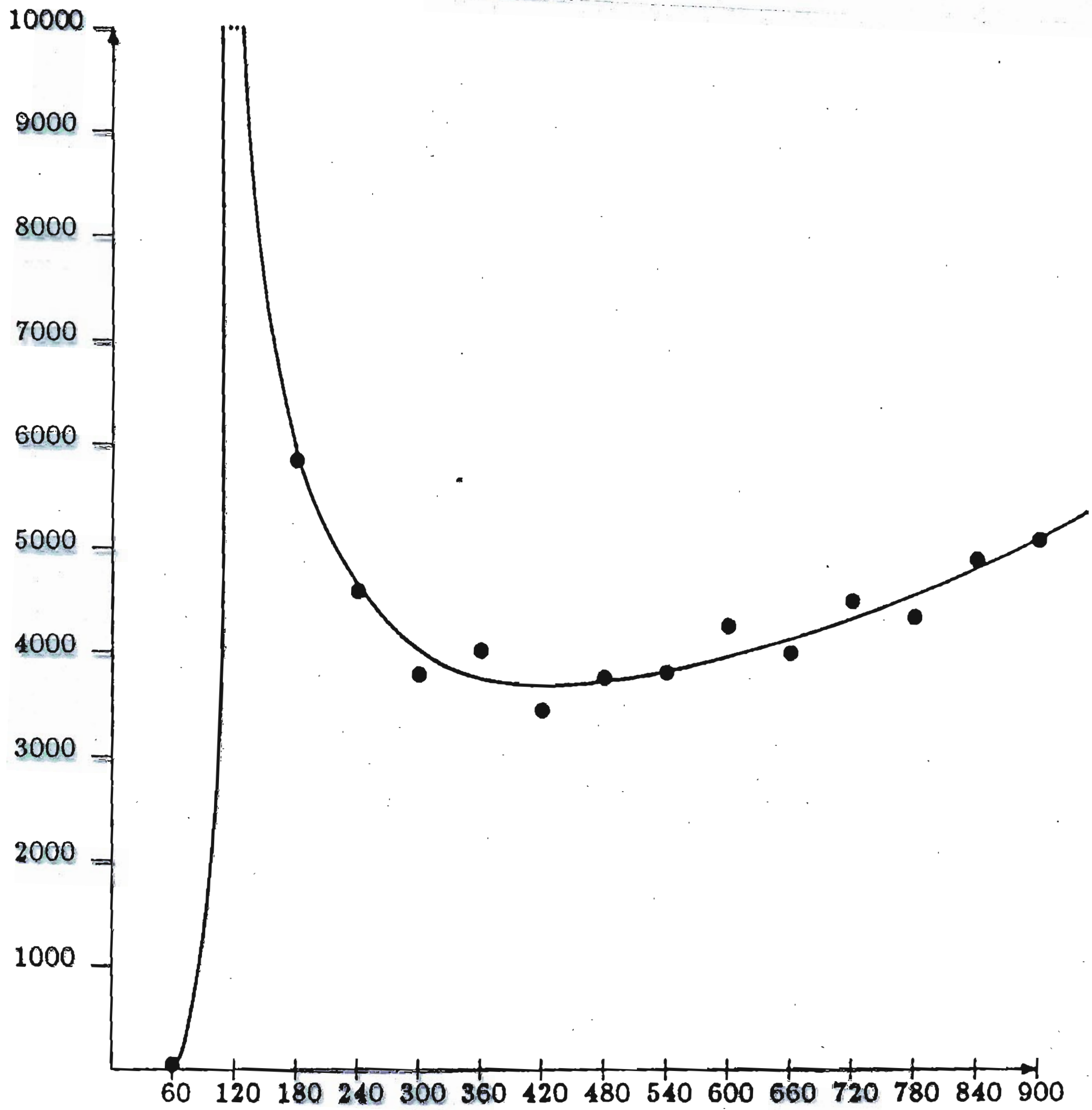


Fig. 2. $p=0.100$, $k=4$.

Classical Metropolis Algorithm

Origin - statistical mechanics

In equilibrium probability $\pi(S)$ that a system is in state S is proportional to

$$e^{-\beta E(S)} \qquad \beta = 1/kT > 0$$

where $E(S)$ is the energy.

Thus

$$\pi(S) = e^{-\beta E(S)} / Z$$

where Z is the "unknown" partition function.

Metropolis et al (1953) proposed

Construct ergodic Markov chain on the set of states s.t. limiting distribution is $\pi(S)$.

Method: Find some irreducible $\{Z_n^*\}$ s.t.

$$\Pr\{Z_n^* = S_j \mid Z_n = S_i\} = \Pr\{Z_n^* = S_i \mid Z_n = S_j\}.$$

Let ΔE = change in energy

$$= E(Z_n^*) - E(Z_n).$$

Define transition mechanism for $\{Z_n\}$ by :

If $\Delta E \leq 0$ then $Z_{n+1} = Z_n^*$

$\Delta E > 0$ then

$$Z_{n+1} = \begin{cases} Z_n^* & \text{with prob } e^{-\beta \Delta(E)} \\ Z_n & \text{with prob } 1 - e^{-\beta \Delta (E} \end{cases}$$

Claim $\pi(S_i) P(S_i \rightsquigarrow S_j) = \pi(S_j) P(S_j \rightsquigarrow S_i)$

in other words reversible

Proof WLOG assume $E(S_j) < E(S_i)$

$$\text{LHS} \propto e^{-\beta E(S_i)} \Pr(Z_n^* = S_j \mid Z_n = S_i) \cdot 1$$

$$\text{RHS} \propto e^{-\beta E(S_j)} \Pr(Z_n^* = S_i \mid Z_n = S_j) \, e^{-\beta \cancel{(\Delta E)}}$$

$$e^{-\beta(E(S_i) - E(S_j))}$$

$\therefore$ LHS = RHS because Z_n^* symmetric

Example Take Ω = all 3-colourings of G

$$\pi(\omega) = \Theta^{\text{bad}(\omega)} / Z$$

Choose irreducible $Q(\omega, \omega')$ by

$$Q(\omega, \omega') = \begin{cases} 0 & \text{unless } d(\omega, \omega') = 1 \\ \frac{1}{n} \cdot \frac{1}{3} & \text{if distance } = 1 \end{cases}$$

Hence

$$\alpha(\omega, \omega') = \begin{cases} 1 & \text{if } \pi(\omega') > \pi(\omega) \\ \dfrac{\pi(\omega)}{\pi(\omega')} & \text{if } \pi(\omega) < \pi(\omega') \end{cases}$$

So if $\Theta < 1$

Move to ω' with probability 1

if $\text{bad}(\omega') < \text{bad}(\omega)$

but only with prob $\dfrac{\Theta^{\text{bad}(\omega')}}{\Theta^{\text{bad}(\omega)}}$

if $\text{bad}(\omega') > \text{bad}(\omega)$.

Convergence of Markov chains

(Total) variation distance between two dist^{ns} π and μ on Ω is

$$d(\pi,\mu) = \|\pi - \mu\| = \frac{1}{2}\sum_{\omega\in\Omega} |\mu(\omega) - \pi(\omega)|$$

$$= \max_{A\subseteq\Omega} |\pi(A) - \mu(A)|$$

Mixing time (function) $\tau^i(\epsilon)$ defined by

$$\tau^i(\epsilon) = \min t : d(P_t^i, \pi) \le \epsilon$$

$$\tau(\epsilon) = \max_{i\in\Omega} \tau^i(\epsilon).$$

Bootstrap Lemmas (Aldous Fill Ch. 2)

1) If t is the first time at which

$$\| P^i_t - \pi \| \leq 1/4$$

then

$$\| P^i_{kt} - \pi \| \leq 2^{-k}$$

for all $k \in \mathbb{N}$.

or

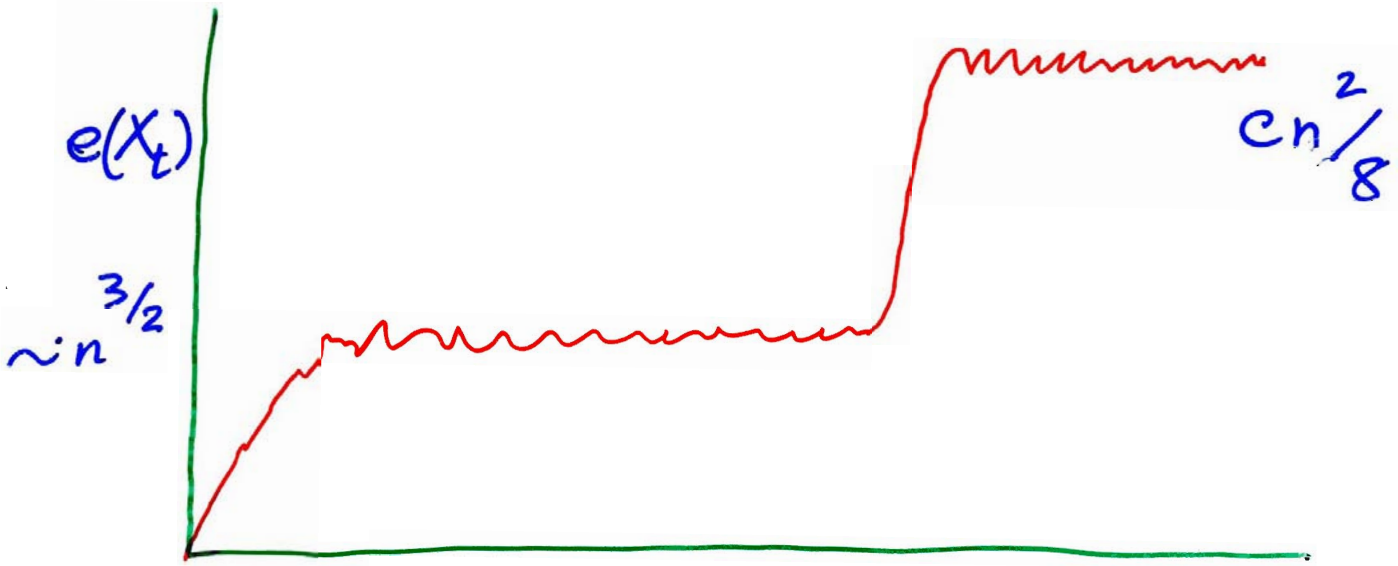
Suppose that Z_t has $\tau(c) \leq T$ for some $c < 1/2$.

Then for any $\epsilon > 0$

$$\tau(\epsilon) \leq \frac{\ln(\epsilon)}{\ln(2c)} T.$$

Random Triangle-Free Graph

Apply simple random walk Markov chain as in random planar case.



$e(X_t) =$ number of edges at time t.

Angelika Steger idea, Thomas Schickinger programme.

Process is **Metastable**