Induction examples
Lecture 5

COSC 242 – Algorithms and Data Structures

Today's outline

1. Proving the sum of consecutive natural numbers
2. Divisible by 3
3. Proving the runtime of Insertion sort
4. More examples

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Recall: Steps of Induction

1. Base case: Verify that $P(1)$ is True.
2. Induction step: Use the assumption that $P(n)$ is True, to prove that $P(n+1)$ is True.

The hypothesis in Step 2, that our statement holds for a particular n, is called the **induction hypothesis**.

To prove the inductive step, we assume the induction hypothesis for n, and then use this assumption to prove that the statement holds for $\mathrm{n}+1$.

Replace n with n+1

1. **Base case**: $P(1)$: $\sum_{k=1}^{n} \frac{1(1+1)}{2} = 1$
2. **Induction step**: We now use the assumption that $P(n)$ is True to show that $P(n+1)$ is also True.

We start by replacing all instances of n in $P(n)$ with n+1.

$$P(n+1)\text{:}\ \sum_{k=1}^{n+1} k = \frac{(n+1)\big((n+1)+1\big)}{2} = \frac{(n+1)(n+2)}{2}$$

Make the left side match the right side

We now take the left side of the equality, and try to make it look like the right side of the equality:

Make this → Look like this

$$P(n+1)\colon \underbrace{\sum_{k=1}^{n+1} k}_{\text{Left side}} = \underbrace{\frac{(n+1)(n+2)}{2}}_{\text{Right side}}$$

Make the LHS match the RHS

$$P(n+1): \underbrace{\sum_{k=1}^{n+1} k}_{\text{Left side}} = \left(\sum_{k=1}^{n} k\right) + (n+1)$$

We write this as sum of all numbers up to n, then add n+1, the next number in the sequence.

Use the assumption that P(n) is True

We now use the assumption that $P(n)$ is True, by replacing $\sum_{k=1}^{n+1} k$ with $\mathrm{n}(n+1)/2$.

$$P(n+1)\text{:}\ \sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

We now use algebra to show the two sides of the equality are equivalent.

Rearranging the left hand side

$$P(n+1):\ \overbrace{\frac{n(n+1)}{2}+(n+1)}^{\text{Left side}}=\overbrace{\frac{(n+1)(n+2)}{2}}^{\text{Right side}}$$

Left hand side becomes:

$$\frac{n(n+1)}{2}+(n+1)=\frac{n(n+1)+2(n+1)}{2}$$

$$\frac{n^2+n+2n+2}{2}=\frac{(n+1)(n+2)}{2}$$

Or, $\frac{n(n+1)}{2}+(n+1)=(n+1)\left(\frac{n}{2}+1\right)$

Conclusion: Since both the base case and the inductive step have been proved, by mathematical induction the statement P(n) holds for every natural number n. ■

Key to induction

Our ability to relate $P(n+1)$ to $P(n)$, which permits us to use the **induction hypothesis** that $P(n)$ is True, is the key to our success of using proof by induction.

When considering a proof technique, and you are unable to relate $P(n+1)$ to $P(n)$, then another technique may be needed.

However, if you can relate $P(n+1)$ to $P(n)$, then induction is easier to use than almost any other proof technique.

Confused by the proof?

This Kahn Academy video provides an even gentler introduction to this proof [1].

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Divisible by 3

Lets return to our first example in Lecture 04 that pushed us to examine mathematical induction [L04, Slide #39].

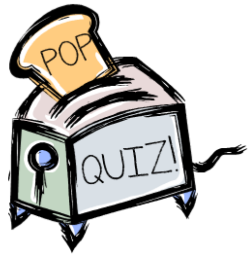
Consider the set $\mathrm{X} \in \mathbb{R}$, where

X = {3, 12, 33, 72, …}

Where X is given by $f(n) = n^3 + 2n$

Are all the elements of X divisible by 3?

Pop quiz 1



Question

Does this example lend itself to proof by induction? Why?

Where X is given by $f(n) = n^3 + 2n$

Are all the elements of X divisible by 3?

Proof

Base case: $f(1) = 1^3 + 2 \cdot 1 = 3$ - True

Inductive step: $P(n+1)$: $f(n+1) = (n+1)^3 + 2 \cdot (n+1)$

$$f(n+1) = (n+1)^3 + 2 \cdot (n+1)$$
$$= (n+1)(n+1)(n+1) + 2n + 2$$

Proof

Base case: $f(1) = 1^3 + 2 \cdot 1 = 3$ - True

Inductive step: $P(n+1)$: $f(n+1) = (n+1)^3 + 2 \cdot (n+1)$

$$
\begin{aligned}
f(n+1) &= (n+1)^3 + 2 \cdot (n+1) \\
&= (n+1)(n+1)(n+1) + 2n + 2 \\
&= (n^2 + 2n + 1)(n+1) + 2n + 2 \\
&= (n^3 + 3n^2 + 3n + 1) + 2n + 2
\end{aligned}
$$

Proof

Base case: $f(1) = 1^3 + 2 \cdot 1 = 3$ - True

Inductive step: $P(n+1)$: $f(n+1) = (n+1)^3 + 2 \cdot (n+1)$

$$
\begin{aligned}
f(n+1) &= (n+1)^3 + 2 \cdot (n+1) \\
&= (n+1)(n+1)(n+1) + 2n + 2 \\
&= (n^2 + 2n + 1)(n+1) + 2n + 2 \\
&= (n^3 + 3n^2 + 3n + 1) + 2n + 2 \\
&= n^3 + 2n + 3n^2 + 3n + 3 \\
&= f(n) + 3n^2 + 3n + 3
\end{aligned}
$$

Here's our f(n). Lets use the induction hypothesis, and assume that f(n) is True…

Proof

Base case: $f(1) = 1^3 + 2 \cdot 1 = 3$ - True

Inductive step: $P(n+1)$: $f(n+1) = (n+1)^3 + 2 \cdot (n+1)$

$$
\begin{aligned}
f(n+1) &= (n+1)^3 + 2 \cdot (n+1) \\
&= (n+1)(n+1)(n+1) + 2n + 2 \\
&= (n^2 + 2n + 1)(n+1) + 2n + 2 \\
&= (n^3 + 3n^2 + 3n + 1) + 2n + 2 \\
&= n^3 + 2n + 3n^2 + 3n + 3 \\
&= f(n) + 3n^2 + 3n + 3 \\
&= f(n) + 3 \cdot (n^2 + n + 1)
\end{aligned}
$$

Here's our f(n). Lets use the induction hypothesis, and assume that f(n) is True…

This term will always be divisible by 3.

Proof

Base case: $f(1) = 1^3 + 2 \cdot 1 = 3$ - True

Inductive step: $P(n+1)$: $f(n+1) = (n+1)^3 + 2 \cdot (n+1)$

$$
\begin{aligned}
f(n+1) &= (n+1)^3 + 2 \cdot (n+1) \\
&= (n+1)(n+1)(n+1) + 2n + 2 \\
&= (n^2 + 2n + 1)(n+1) + 2n + 2 \\
&= (n^3 + 3n^2 + 3n + 1) + 2n + 2 \\
&= n^3 + 2n + 3n^2 + 3n + 3 \\
&= f(n) + 3n^2 + 3n + 3 \\
&= f(n) + 3 \cdot (n^2 + n + 1)
\end{aligned}
$$

Here's our f(n). Lets use the induction hypothesis, and assume that f(n) is True…

This term will always be divisible by 3.

Conclusion: Since both the base case and the inductive step have been proved, the statement P(n) is True for n+1, and so the proof is complete. ■

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Insertion sort

In lecture L01, we stated that the run time of insertion sort was roughly indicated by $0 + 1 + 2 + 3 + \cdots + \mathrm{n} - 1 = \frac{n(n-1)}{2} = \frac{n^2-n}{2}$.

Section 2.1 of the text explains how we arrived at that equation.

This example is slightly different to our earlier proofs, as we have a left hand side (LHS) and a right hand side (RHS):

$$l(n) = 1 + 2 + \cdots + n - 1$$

$$r(n) = \frac{n(n-1)}{2}$$

We want to show that $\mathrm{l(n)} = \mathrm{r(n)}$ for all n ≥ 1

Class challenge 1

How did we arrive at $\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$?

Solve insertion sort with induction

$l(n) = 1 + 2 + \cdots + n - 1$
$r(n) = \frac{n(n-1)}{2}$

Base case: $l(1) = 0 = r(1) = \frac{1(1-1)}{2} = 0$

Solve insertion sort with induction

$l(n) = 1 + 2 + \cdots + n - 1$
$r(n) = \frac{n(n-1)}{2}$

Base case: $l(1) = 0 = r(1) = \frac{1(1-1)}{2} = 0$

Inductive step: assume $l(n) = r(n)$, which is our $P(n)$ induction hypothesis, and use it to show that $l(n+1) = r(n+1)$

Solve insertion sort with induction

$l(n) = 1 + 2 + \cdots + n - 1$
$r(n) = \frac{n(n-1)}{2}$

Base case: $l(1) = 0 = r(1) = \frac{1(1-1)}{2} = 0$

Inductive step: assume $l(n) = r(n)$, which is our $P(n)$ induction hypothesis, and use it to show that $l(n+1) = r(n+1)$

$$l(n+1) = 1 + 2 + \cdots + n - 1 + n$$
$$= l(n) + n = r(n) + n$$

Solve insertion sort with induction

$l(n) = 1 + 2 + \cdots + n - 1$

$r(n) = \frac{n(n-1)}{2}$

Base case: $l(1) = 0 = r(1) = \frac{1(1-1)}{2} = 0$

Inductive step: assume $l(n) = r(n)$, which is our $P(n)$ induction hypothesis, and use it to show that $l(n+1) = r(n+1)$

$$l(n+1) = 1 + 2 + \cdots + n - 1 + n$$
$$= l(n) + n = r(n) + n$$
$$r(n+1) = (n+1)\big((n+1) - 1\big)/2$$
$$= (n+1)(n)/2$$

Lets try and make this look like our RHS(n) $= \frac{n^2 - n}{2}$

Solve insertion sort with induction

$l(n) = 1 + 2 + \cdots + n - 1$

$r(n) = \frac{n(n-1)}{2}$

Base case: $l(1) = 0 = r(1) = \frac{1(1-1)}{2} = 0$

Inductive step: assume $l(n) = r(n)$, which is our $P(n)$ induction hypothesis, and use it to show that $l(n+1) = r(n+1)$

$$l(n+1) = 1 + 2 + \cdots + n - 1 + n$$
$$= l(n) + n = r(n) + n$$
$$r(n+1) = (n+1)\big((n+1) - 1\big)/2$$
$$= (n+1)(n)/2$$
$$= (n^2 + n)/2$$
$$= (n^2 - n + 2n)/2$$
$$= (n^2 - n)/2 + 2n/2$$
$$= r(n) + n = l(n) + n$$

Lets try and make this look like our RHS(n) = $\frac{n^2-n}{2}$

Conclusion: By P.M.I, the statement P(n) is True for n+1, and so the proof is complete. ■

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Recursively defined sets

We can use induction proofs on any set that is recursively defined.

For example, we can define the set of natural numbers, $\mathbb{N}$ as:

Base case: $0 \in \mathbb{N}$

Inductive step: if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$

Similarly, the set of prime numbers, $\mathbb{P}$ as:

Base case: $2 \in \mathbb{P}$

Inductive step: $p \in \mathbb{P}$, if and only if, $p \neq cn$ for some constant c, and then $n < p$

Prove that $n^2 = O(2^n)$

Prove that $n^2 = O(2^n)$.

We begin by stating the definition of Big-O:

Show that there exists some c, n_0 such that $n^2 \leq c \cdot 2^n$ for all n ≥ n_0

Lets choose appropriate values of c, n_0

c = 1, n_0 = 4

Lets rewrite as a more traditional induction proof:

Prove that $n^2 = O(2^n)$

Prove that $n^2 \leq 2^n$ for all n ≥ 4

Base case: $4^2 = 16 \leq 2^4 = 16$

Induction hypothesis: $2^n \geq n^2$ for some n ≥ 4

Inductive step: $2^{n+1} \geq (n+1)^2 = n^2 + 2n + 1$

Prove that $n^2 = O(2^n)$

Prove that $n^2 \leq 2^n$ for all n ≥ 4

Base case: $4^2 = 16 \leq 2^4 = 16$

Induction hypothesis: $2^n \geq n^2$ for some n ≥ 4

Inductive step: $2^{n+1} \geq (n+1)^2 = n^2 + 2n + 1$

$$2^{n+1} = 2 \cdot 2^n$$
$$= 2^n + 2^n$$
$$\geq n^2 + n^2$$

Lets use our induction hypothesis here, that $2^n > n^2$

Prove that $n^2 = O(2^n)$

Prove that $n^2 \leq 2^n$ for all n ≥ 4

Base case: $4^2 = 16 \leq 2^4 = 16$

Induction hypothesis: $2^n \geq n^2$ for some n ≥ 4

Inductive step: $2^{n+1} \geq (n+1)^2 = n^2 + 2n + 1$

$$
\begin{aligned}
2^{n+1} &= 2 \cdot 2^n \\
&= 2^n + 2^n \\
&\geq n^2 + n^2 \\
&\geq n^2 + n \cdot n \\
&\geq n^2 + 4 \cdot n
\end{aligned}
$$

Lets use our induction hypothesis here, that $2^n > n^2$

Lets use the second part of our induction hypothesis, n ≥ 4

Prove that $n^2 = O(2^n)$

Prove that $n^2 \leq 2^n$ for all n ≥ 4

Base case: $4^2 = 16 \leq 2^4 = 16$

Induction hypothesis: $2^n \geq n^2$ for some n ≥ 4

Inductive step: $2^{n+1} \geq (n+1)^2 = n^2 + 2n + 1$

$$
\begin{aligned}
2^{n+1} &= 2 \cdot 2^n \\
&= 2^n + 2^n \\
&\geq n^2 + n^2 \\
&\geq n^2 + n \cdot n \\
&\geq n^2 + 4 \cdot n \\
&\geq n^2 + 2 \cdot n + 2 \cdot n \\
&\geq n^2 + 2 \cdot n + 8 \\
&\geq n^2 + 2 \cdot n + 1
\end{aligned}
$$

Lets use our induction hypothesis here, that $2^n > n^2$

Lets use the second part of our induction hypothesis, n ≥ 4

Lets use the second part of our induction hypothesis, n ≥ 4

Prove that $n^2 = O(2^n)$

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$$
\begin{aligned}
2^{n+1} &= 2 \cdot 2^n \\
&= 2^n + 2^n \\
&\geq n^2 + n^2 \\
&\geq n^2 + n \cdot n \\
&\geq n^2 + 4 \cdot n \\
&\geq n^2 + 2 \cdot n + 2 \cdot n \\
&\geq n^2 + 2 \cdot n + 8 \\
&\geq n^2 + 2 \cdot n + 1
\end{aligned}
$$

Lets use our induction hypothesis here, that $2^n > n^2$

Lets use the second part of our induction hypothesis, n ≥ 4

Lets use the second part of our induction hypothesis, n ≥ 4

Conclusion: Since both the base case and the inductive step have been proved, the statement P(n) is True for n+1, for all n > 4. ■

Class challenge 2

Prove that $2^n = O(n!)$, for some c, n_0, such that $2^n < n!$ for all $n > n_0$

Helpful resources on proofs

The textbook does not cover mathematical proofs. Instead, you may find the following resources helpful:

Free resources

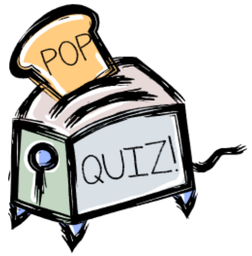
- Data Structures & Algorithm Analysis, C. A. Shaffer, 2013, Dover. [link]
- Mathematical Reasoning: Writing and Proof, T. Sundstrom, 2020, Grand Valley State University. [link]
- Proofs and Mathematical Reasoning, A. Stefanowicz, 2014, University of Birmingham. [link]
- American Institute of Mathematics. [link]

Paid resources

- How to read and do proofs, D. Solow, 2013, Wiley.

Solutions

Pop quiz 1



Question

Does this example lend itself to proof by induction? Why?

Where X is given by $f(n) = n^3 + 2n$

Are all the elements of X divisible by 3?

Answer

For every integer $n \geq 1$, ~~*"something happens".*~~

"f(n) is divisible by 3"

Class challenge 1

How did we arrive at $\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$?

Lets write out our series going forwards, and backwards:

$1 \qquad + \; 2 \qquad + \cdots + n - 1$

$(n-1) + (n-2) + \cdots + 1$

Lets now add the two series together:

$1 + (n-1) + 2 + (n-2) + \cdots + 1$ =

$(n) + (n) + \cdots + (n)$

We now have n-1 many n's. →$n * (n-1)$. But since we added two series together, we need to divide our result by 2, producing: $\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$

Class challenge 2

Prove that $2^n \leq n!$ for all n ≥ 4

Base case: $2^4 = 16 \leq 4! = 4 * 3 * 2 * 1 = 24$

Induction hypothesis: $2^n \leq n!$ for some n ≥ 4

Inductive step: $2^{n+1} \leq \ (n+1)!$
$(n+1)2^n \leq (n+1)n!$
$(n+1)2^n \leq (n+1)!$

Lets multiple both sides of inequality by (n+1)

Since n + 1 > 2, then $(n+1)2^n > 2 \cdot 2^n = 2^{n+1}$

Therefore: $2^{n+1} \leq \ (n+1)!$ for all n ≥ 4

Conclusion: Since both the base case and the inductive step have been proved, the statement P(n) is True for n+1, for all n > 4. ■

References and attributions

1. https://www.khanacademy.org/math/algebra-home/alg-series-and-induction/alg-induction/v/proof-by-induction

Attributions

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