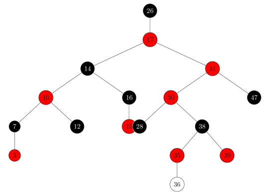

Red-Black Trees 2 - Insertion
Lecture 16

COSC 242 – Algorithms and Data Structures

UNIVERSITY of OTAGO
Te Whare Wānanga o Otāgo
NEW ZEALAND

Today’s outline

1. Rotations
2. Insertion
3. Insert Fixup
4. Cases
5. Examples

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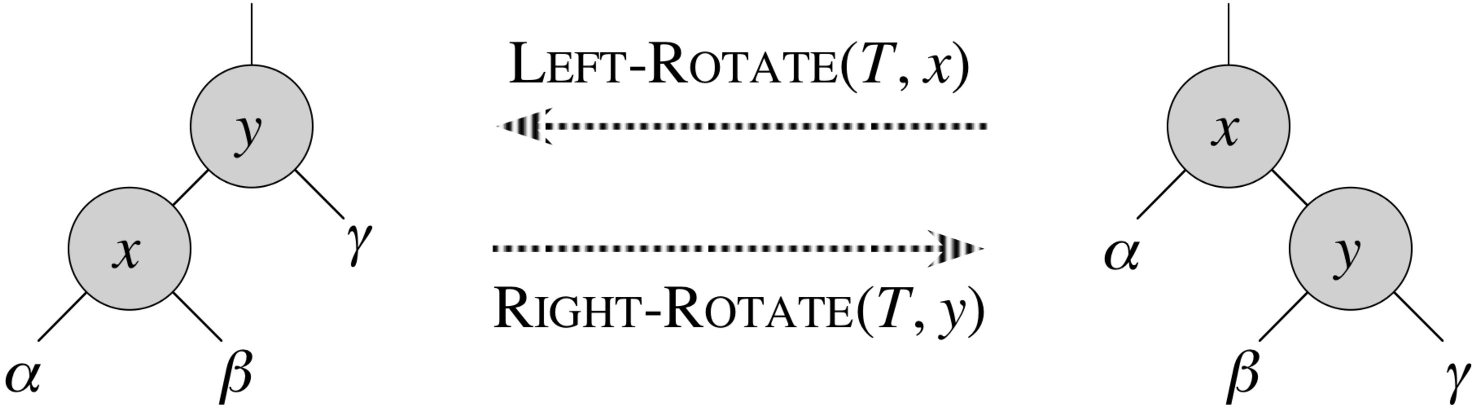
Rotations

The operations insert and delete when run on an RBT with n nodes takes O(log n) time. Because these operations modify the RBT, the result may violate the RBT properties.

Rotation provides efficient rebuilding to maintain these properties.



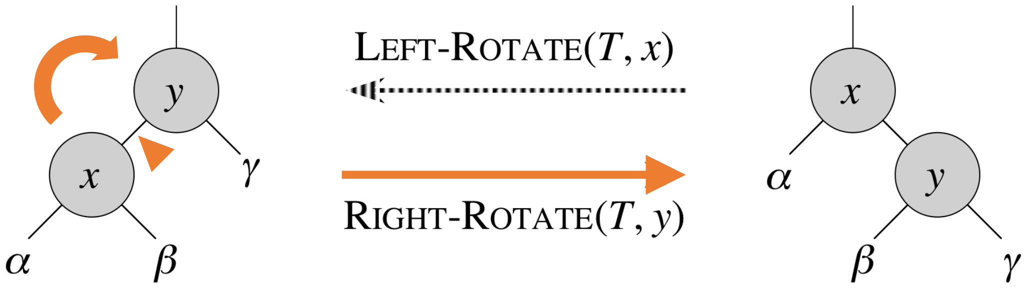
The letters $\alpha, \beta, and\ \gamma$ represent three arbitrary subtrees.

Rotations

Rotations work by updating the pointer structure of the tree. When do do a right-rotation on node *y*, we assume that its left child *x* is not *T.nil*. *y* may be any node in the tree whose left child is not *T.nil*.

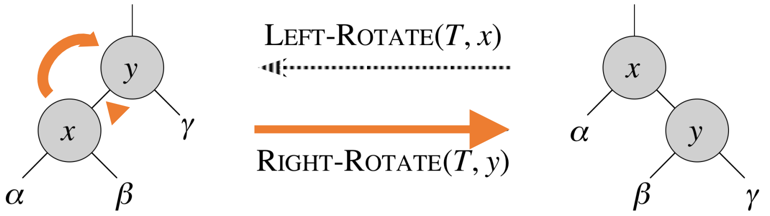
Right-rotation

- "pivots" around the edge from *x* to *y*.
- Makes *x* the new root of the subtree
- *y* becomes *x*'s right child
- *x*'s right child becomes *y*'s left child.



Right rotation


```
procedure RBT_Rotate_Right(x)
    y = x→parent                // set y
    y→left = x→right            // turn x's right subtree β into y's left subtree
    x→right = y                 // set y as x's right subtree
    x→parent = y→parent         // x's parent becomes y's parent
    if y→parent == NILL then
        root = x                // y was root, make x tree root
    else
        y→parent→[left or right] = x // update y's R or L child to point to x
    end if
    y→parent = x                // y's parent is now x
end procedure
```

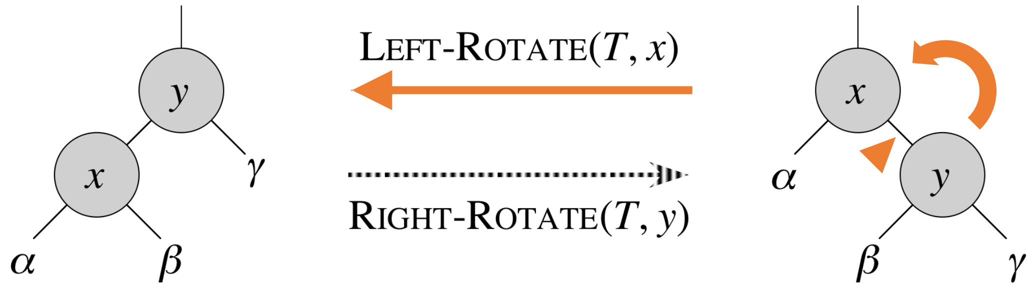


Rotations

When do do a left-rotation on node *x*, we assume that its right child *y* is not *nil*. *x* may be any node in the tree whose right child is not *nil*.

Left-rotation

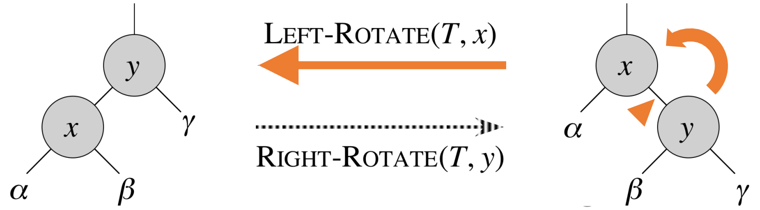
- “pivots” around the edge from *x* to *y*.
- Makes *y* the new root of the subtree
- *x* becomes *y*’s left child
- *y*’s left child becomes *x*’s right child.



The pseudocode for left rotate is symmetric: exchange right with left everywhere.

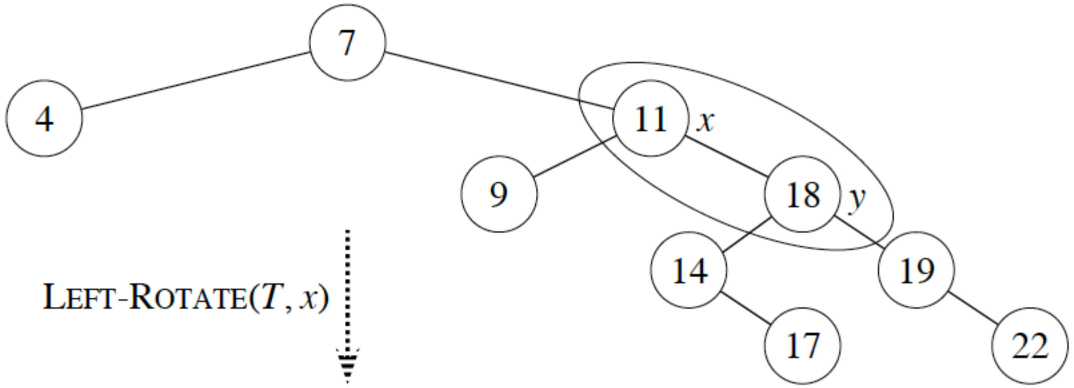
Left rotation

```
procedure RBT_Rotate_Left(x)
    y = x→right                 // set y
    x→right = y→left            // turn y’s left subtree β into x’s right subtree
    y→left = x                  // set x as y’s left subtree
    y→parent = x→parent         // y’s parent becomes x’s parent
    if x→parent == NILL then
        root = y                // x was root, make y tree root
    else
        x→parent→[left or right] = y // update x’s R or L child to point to y
    end if
    x→parent = y                // x’s parent is now y
end procedure
```



Class challenge

Perform an RBT_Rotate_left(T, x) operation:



Node colours omitted for convenience.

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Insertion

The basic algorithm for inserting a node into an RBT is:

```
procedure RBT_Insert(T, x)
    BST_insert(T, x)
    x.colour = RED
    if x→parent == RED then        // Violation of property 4
        RBT_Insert_Fixup(T, x)
    end if
end procedure
```

Insertion

By colouring *x* red, we may violate property 4 that says red nodes have black children. Think of *x* as the problem node. We call `RBT_Insert_Fixup` to restore red-black properties.

All fixups push the problem back up the tree, so `RBT_Insert_Fixup` needs to traverse the tree upwards until either there is no problem anymore, or we reach the root of the tree.

This can be done recursively or iteratively.

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```
procedure RBT_Insert_Fixup(T, z)
    while z→parent.colour == RED
        if z→parent == z→parent→parent→left      // z’s parent is a left-child
            y = z→parent→parent→right            // set y to z’s “uncle”
            if y→colour == RED
                z→parent.colour = BLACK          // case 1
                y.colour = BLACK                 // case 1
                z→parent→parent = RED            // case 1
                z = z→parent→parent              // case 1
            else
                if z = z→parent→right
                    z = z→parent                 // case 2
                    RBT_Rotate_Left(T, z)        // case 2
                end if
            z→parent.colour = BLACK              // case 3
            z→parent→parent = RED                // case 3
            RBT_Rotate_Right(T, z→parent→parent) // case 3
        else ... (same as then clause, with ”right” and “left” exchanged)
    T→root.colour = BLACK
```

Fixup procedure

To understand Fixup, we will break our investigation of the pseudocode into three major steps:

1. We will examine what violates of RBT properties are introduced by RBT_Insert.
2. We will consider the overall goal of the while loop lines 1-17.
3. We will explore each of the three cases.

Property violations

1. Every node is either red or black.
2. The root is black.
3. Every leaf (nil/null) is black.
4. If a node is red, then both its children are black*.
5. For each node, all paths from the node to leaves contain the same number of black nodes.

RBT properties upon entering Fixup

1. Satisfied, as *z* is red.
2. Violated if *z* is the root.
3. Satisfied, as both children of new node are *T.nil*
4. Violated if parent is red, as *z* is also red.
5. Satisfied, as *z* replaces black sentinel, and node *z* is red, with two black sentinel children.

While loop

The **while** loop in lines 1–15 maintains the following three-part invariant at the start of each iteration of the loop:

a) Node *z* is red

b) If *z*→`parent` is the root, then *z*→`parent` is black

c) If the tree violates any of the RBT properties, then it violates at most one of them, which is either property 2 or property 4.

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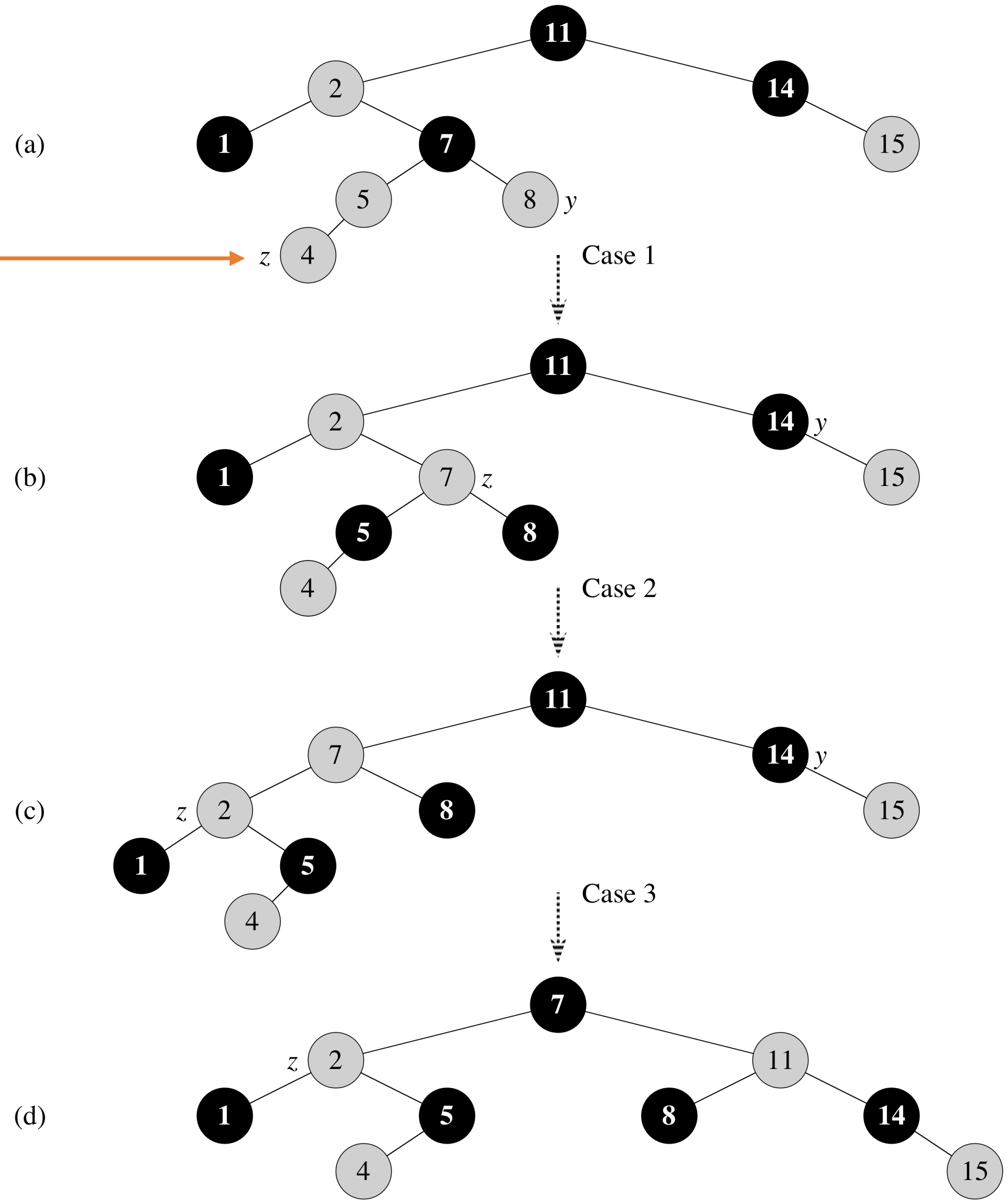
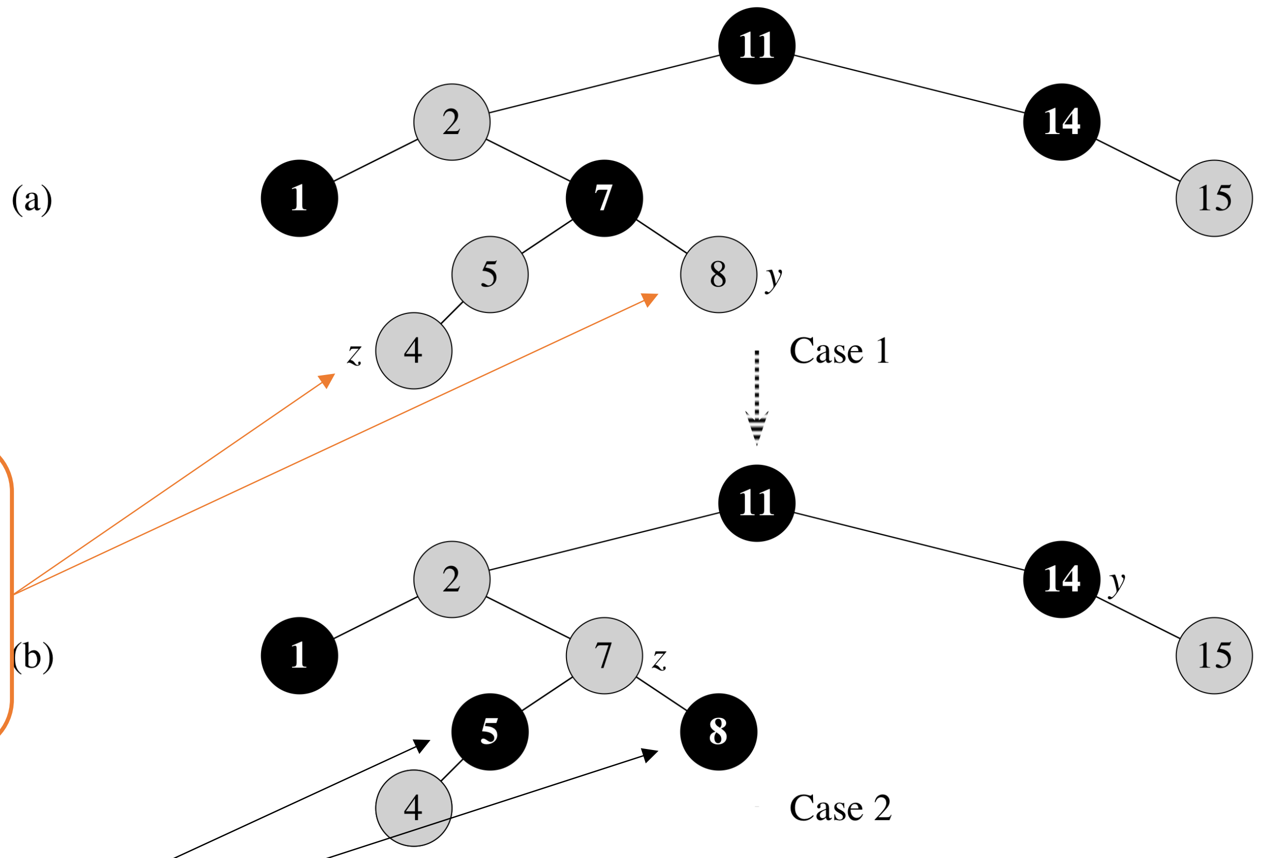
Example

Insert 4



Case 1 violation

```
procedure RBT_Insert_Fixup(T, z)
1:    while z→parent.colour == RED
2:        if z→parent == z→parent→parent→left
3:            y = z→parent→parent→right
4:            if y→colour == RED
5:                z→parent.colour = BLACK // case 1
6:                y.colour = BLACK        // case 1
7:                z→parent→parent = RED   // case 1
8:                z = z→parent→parent     // case 1
9:            else
```

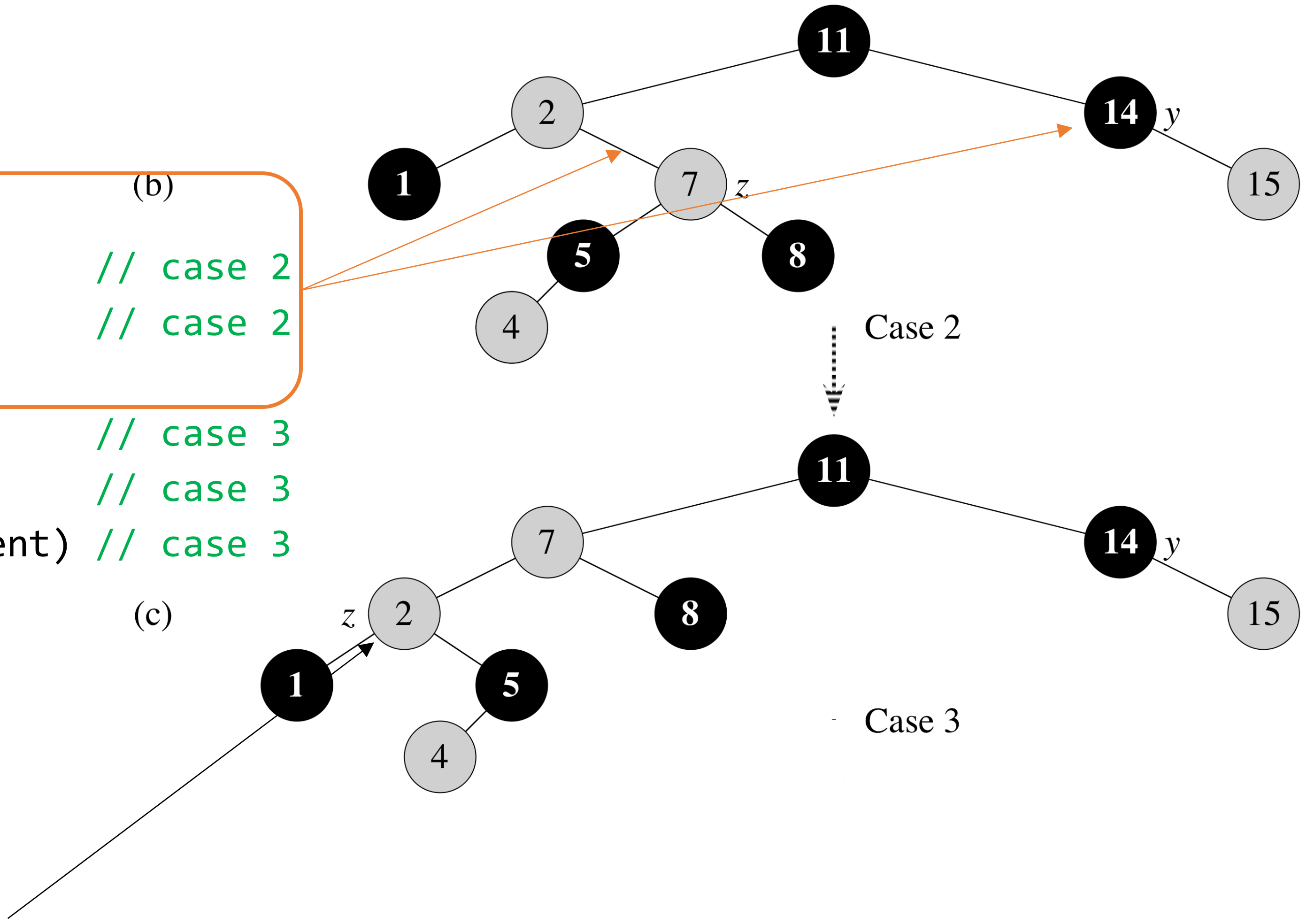


Case 1: z's uncle, y, is red.

- L5-6: Colour z's parent and uncle black
- L7: Colour z's grandparent black
- L8: Z now points to z's grandparent

Case 2 violation

```
9:          else
10:             if z = z→parent→right
11:                 z = z→parent                    // case 2
12:                 RBT_Rotate_Left(T, z)           // case 2
13:             end if
14:         z→parent.colour = BLACK                 // case 3
15:         z→parent→parent = RED                   // case 3
16:         RBT_Rotate_Right(T, z→parent→parent) // case 3
17:     else ...
18  T→root.colour = BLACK
```

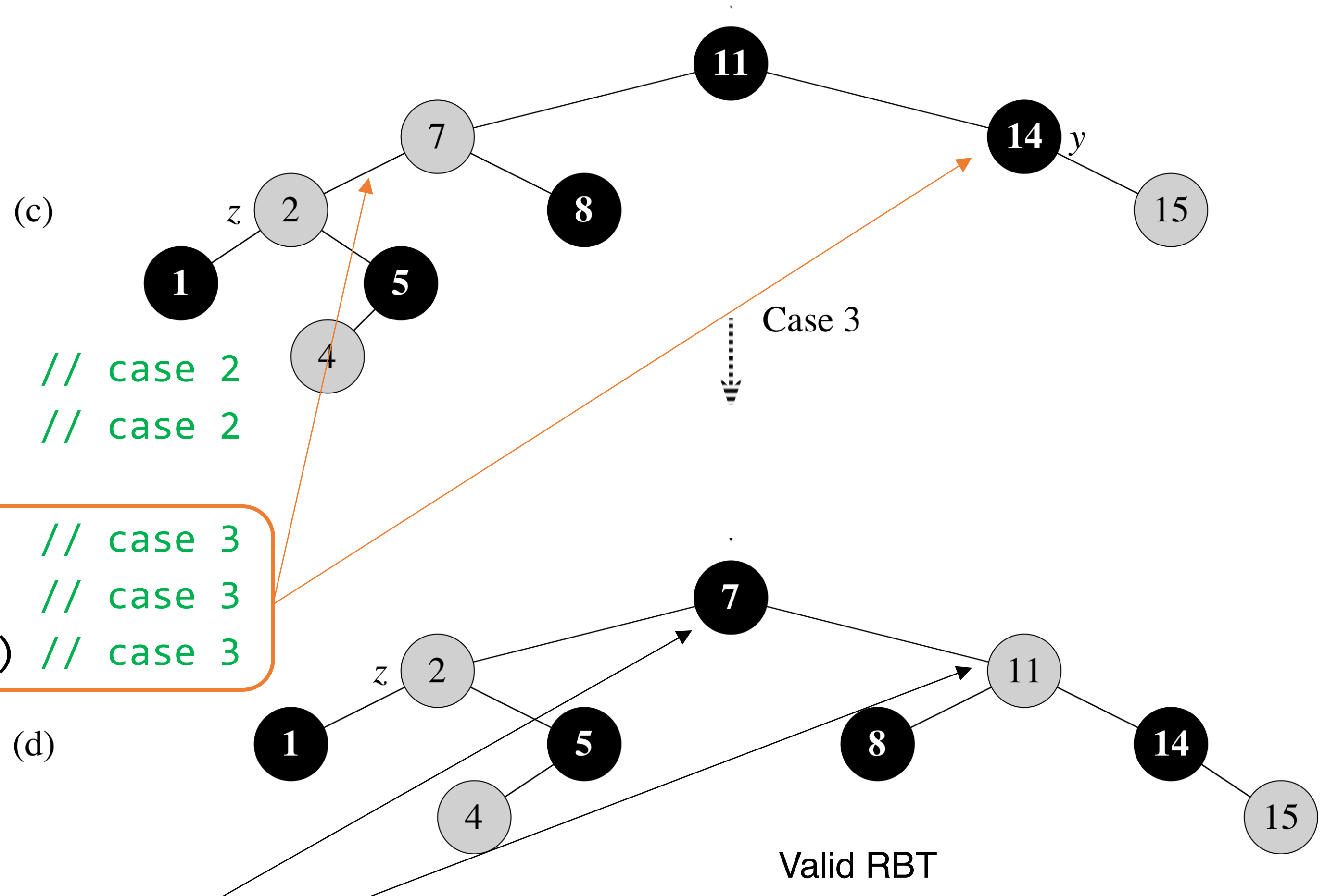


Case 2: z's uncle y is black and z is a right child.

- L11: z now points to z's parent
- L12: Left rotate on z (i.e. old z's parent)

Case 3 violation

```
9:              else
10:                 if z = z→parent→right
11:                     z = z→parent                    // case 2
12:                     RBT_Rotate_Left(T, z)           // case 2
13:                 end if
14:             z→parent.colour = BLACK                 // case 3
15:             z→parent→parent = RED                   // case 3
16:             RBT_Rotate_Right(T, z→parent→parent)    // case 3
17:         else ...
18      T→root.colour = BLACK
```



Case 3 always falls through from case 2

Case 3: z's uncle y is black and z is a left child.

- L14: Colour z's parent black
- L15: Colour z's grandparent red
- L16: Right rotate on z's grandparent

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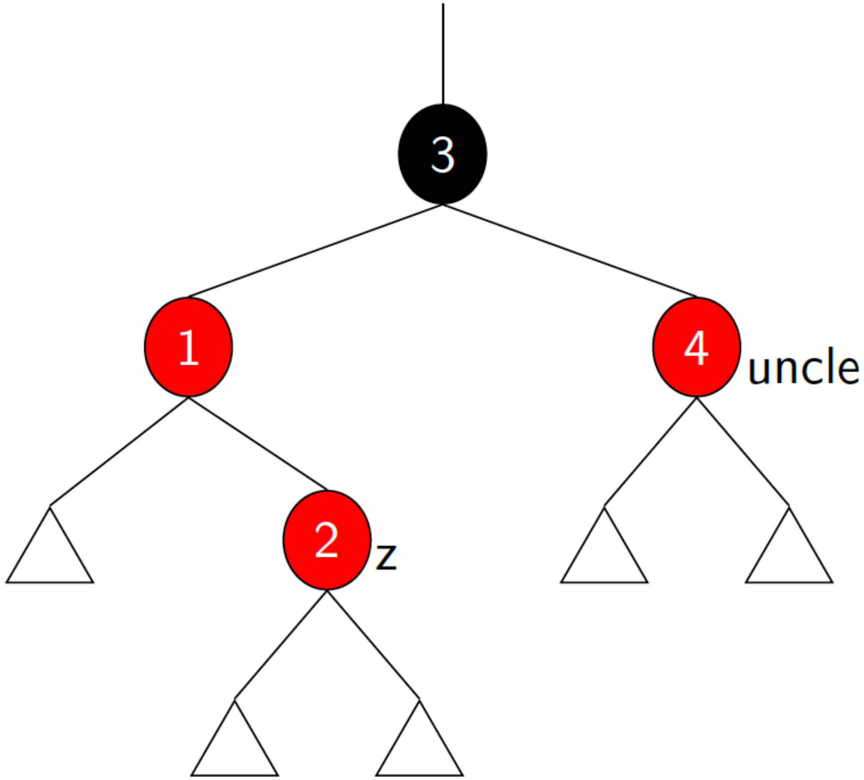
Case 1 violation

Insert: 3, 1, 4, 2

Initial:

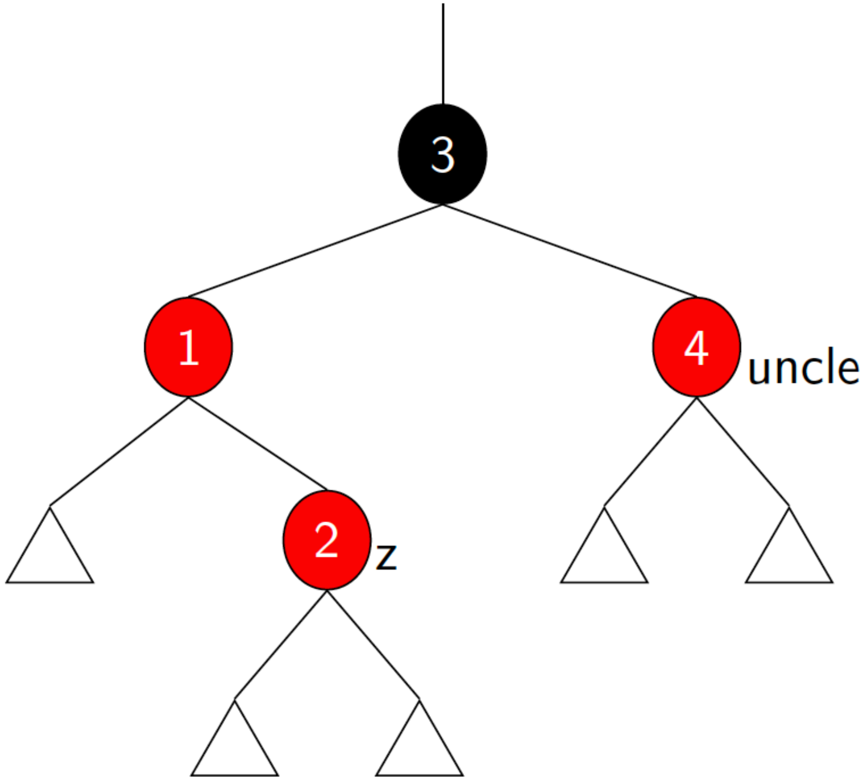


Case 1 violation

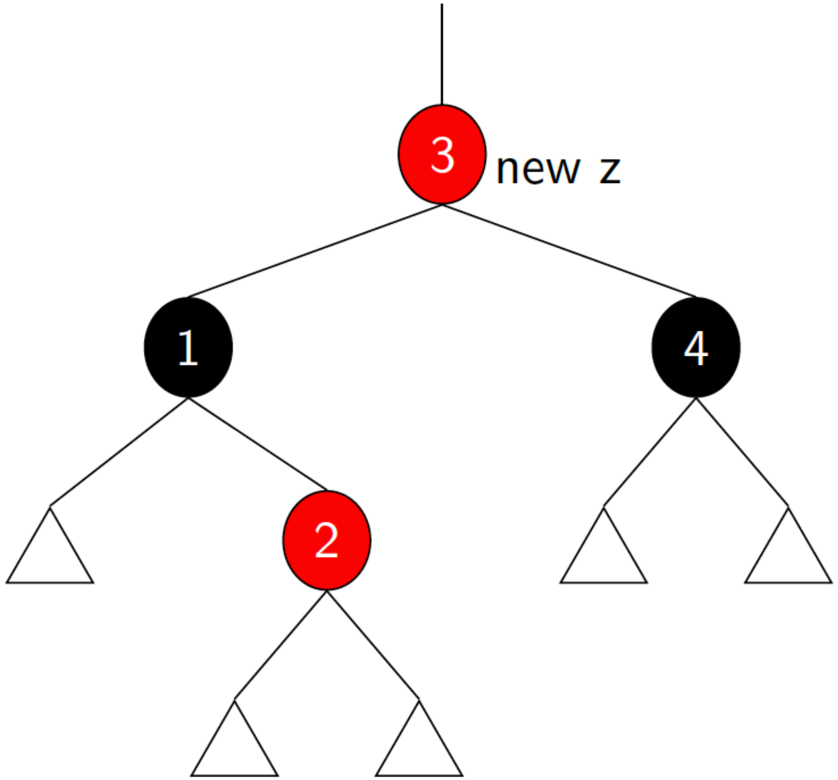
Case 1: *z*'s uncle is red.
It might cause a violation further up the tree.

Insert: 3, 1, 4, 2

Initial:

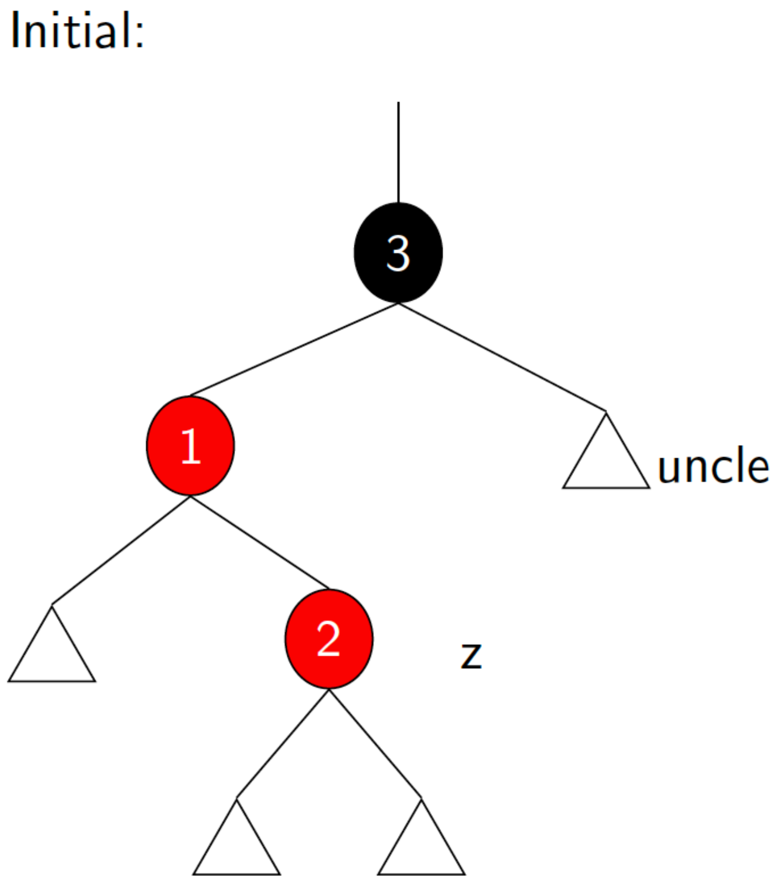


Fixup:



Case 2 violation

Insert: 3, 1, 2

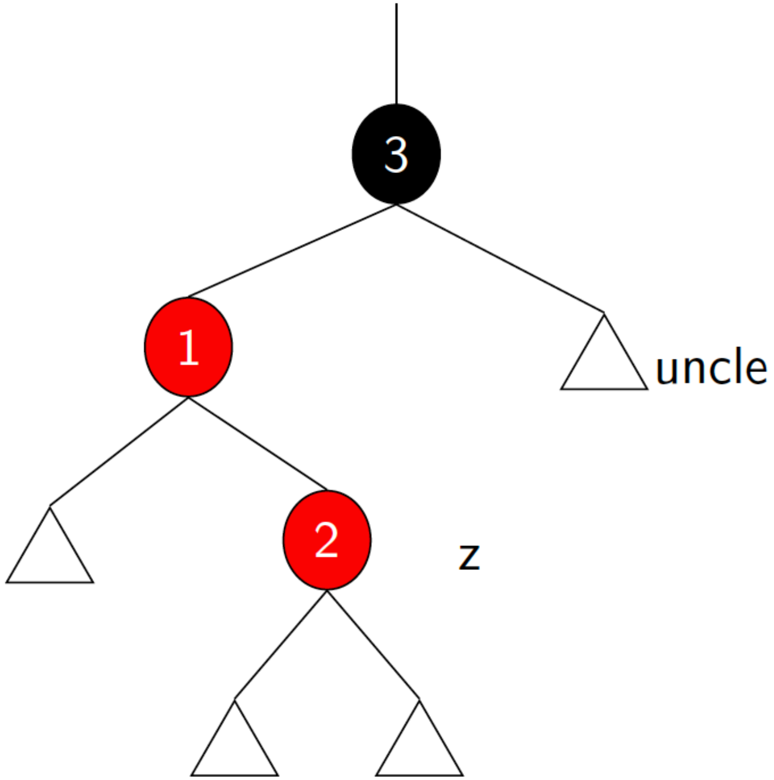


Case 2 violation

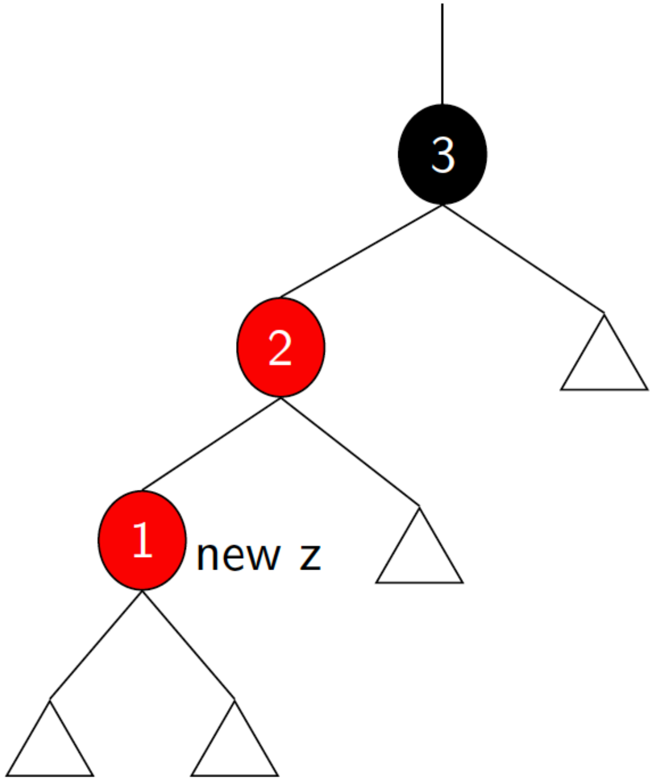
Case 2: z's uncle is black and z is a right child.

Insert: 3, 1, 2

Initial:



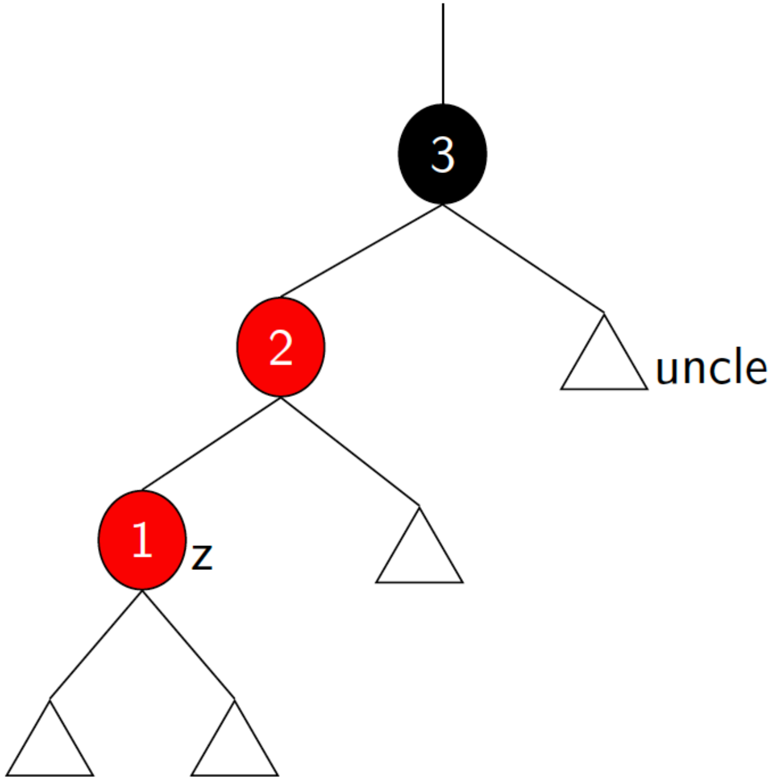
Fixup:



Case 3 violation

Following on from case 2 violation…

Initial:

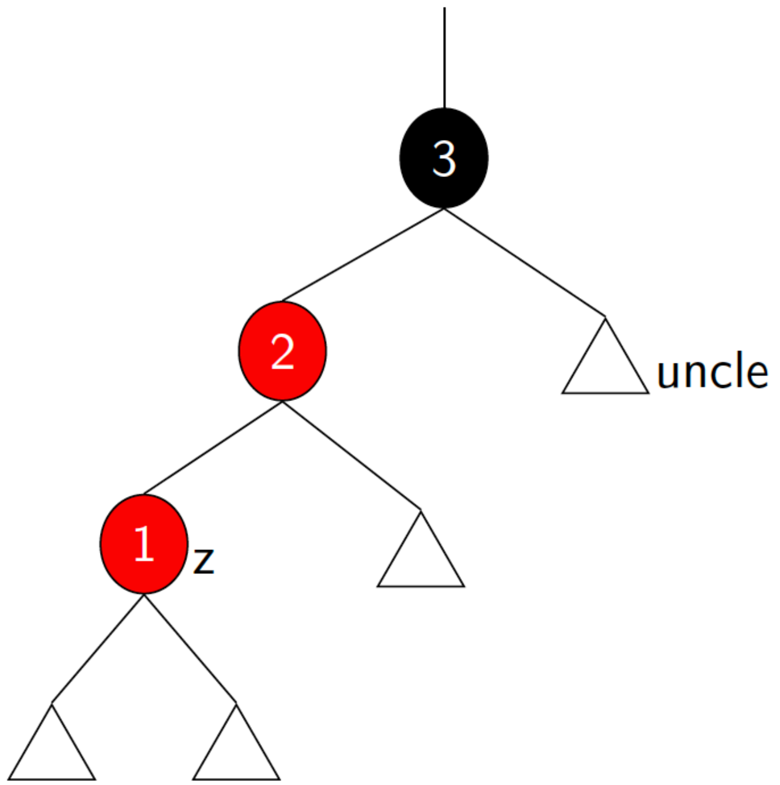


Case 3 violation

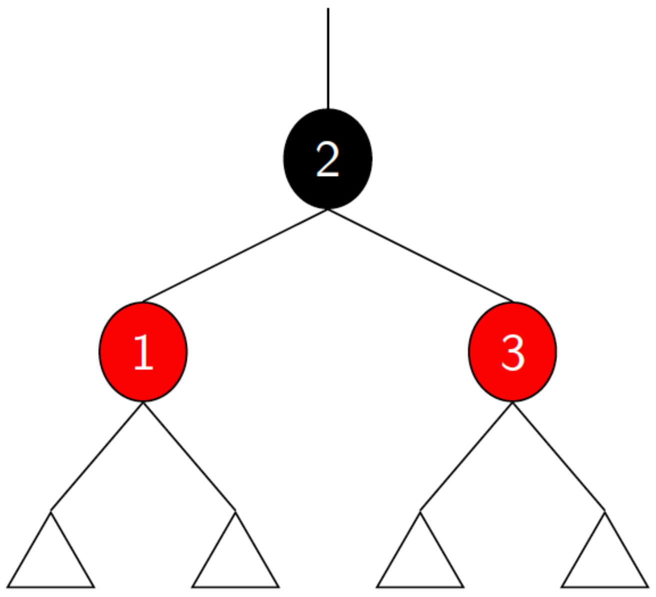
Case 3: z's uncle is black and z is a left child.

Following on from case 2 violation…

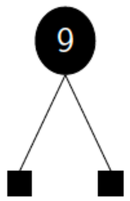
Initial:



Fixup:



Example Insertions (9, 8, 6, 3, 5)



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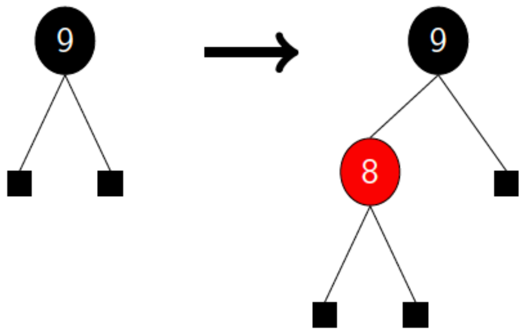

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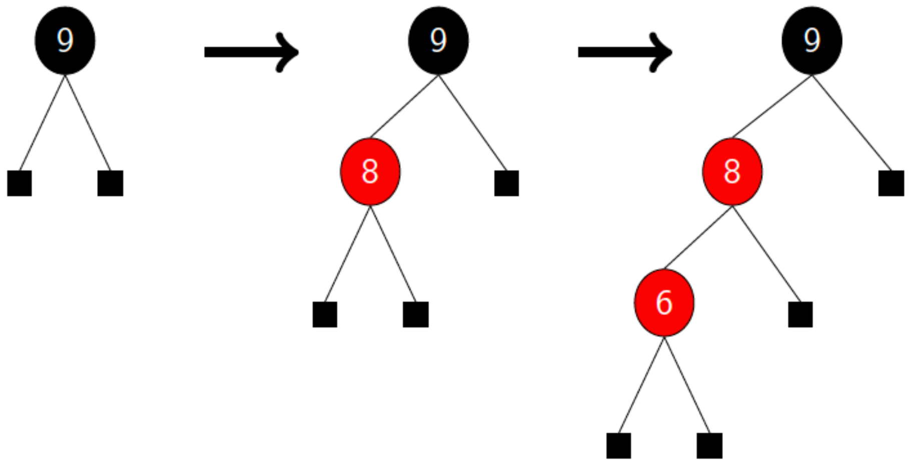
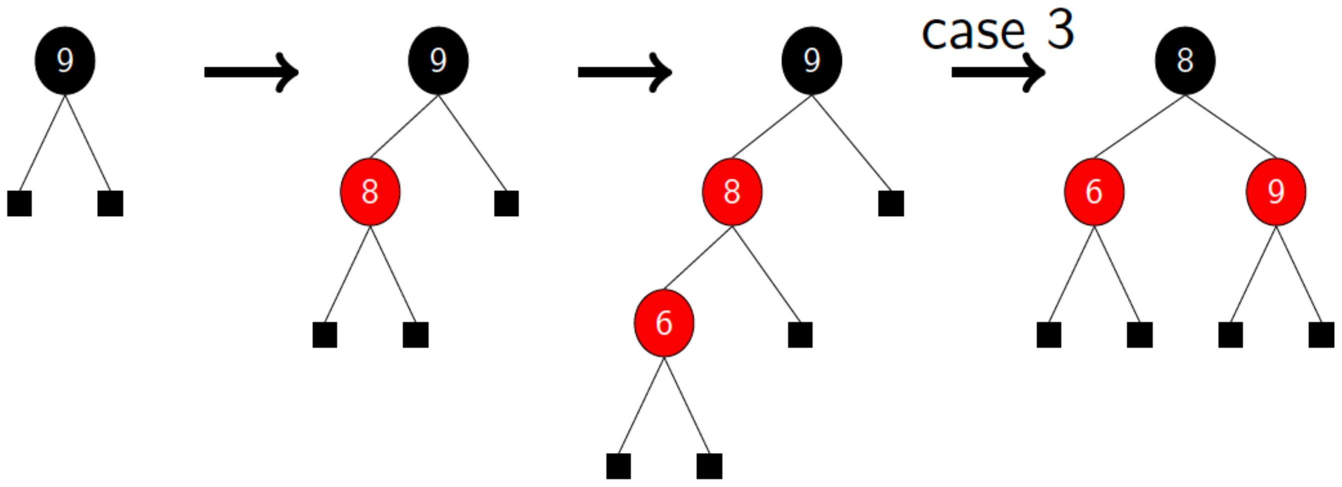
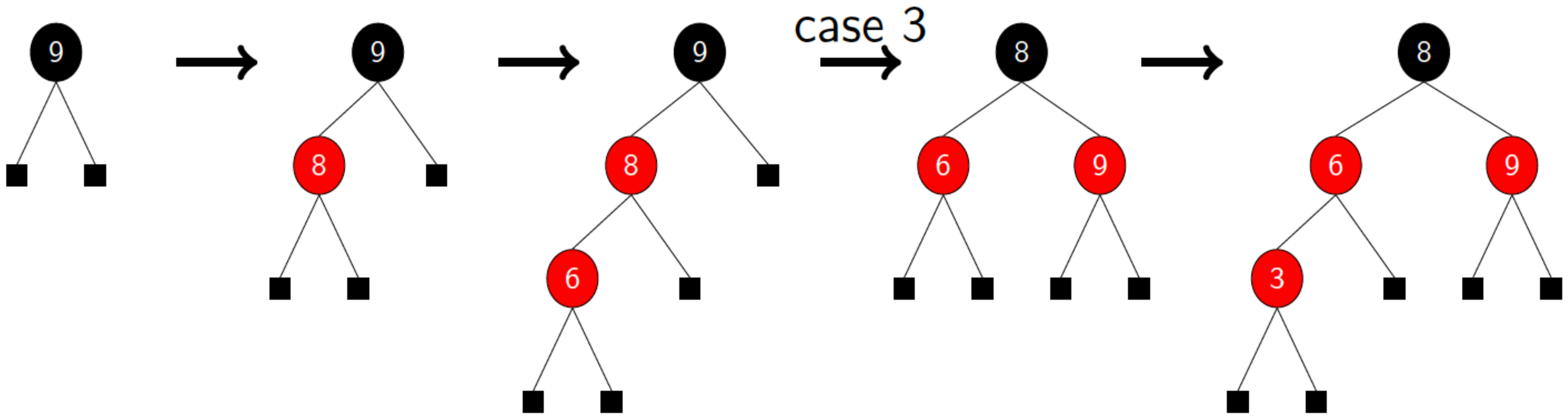


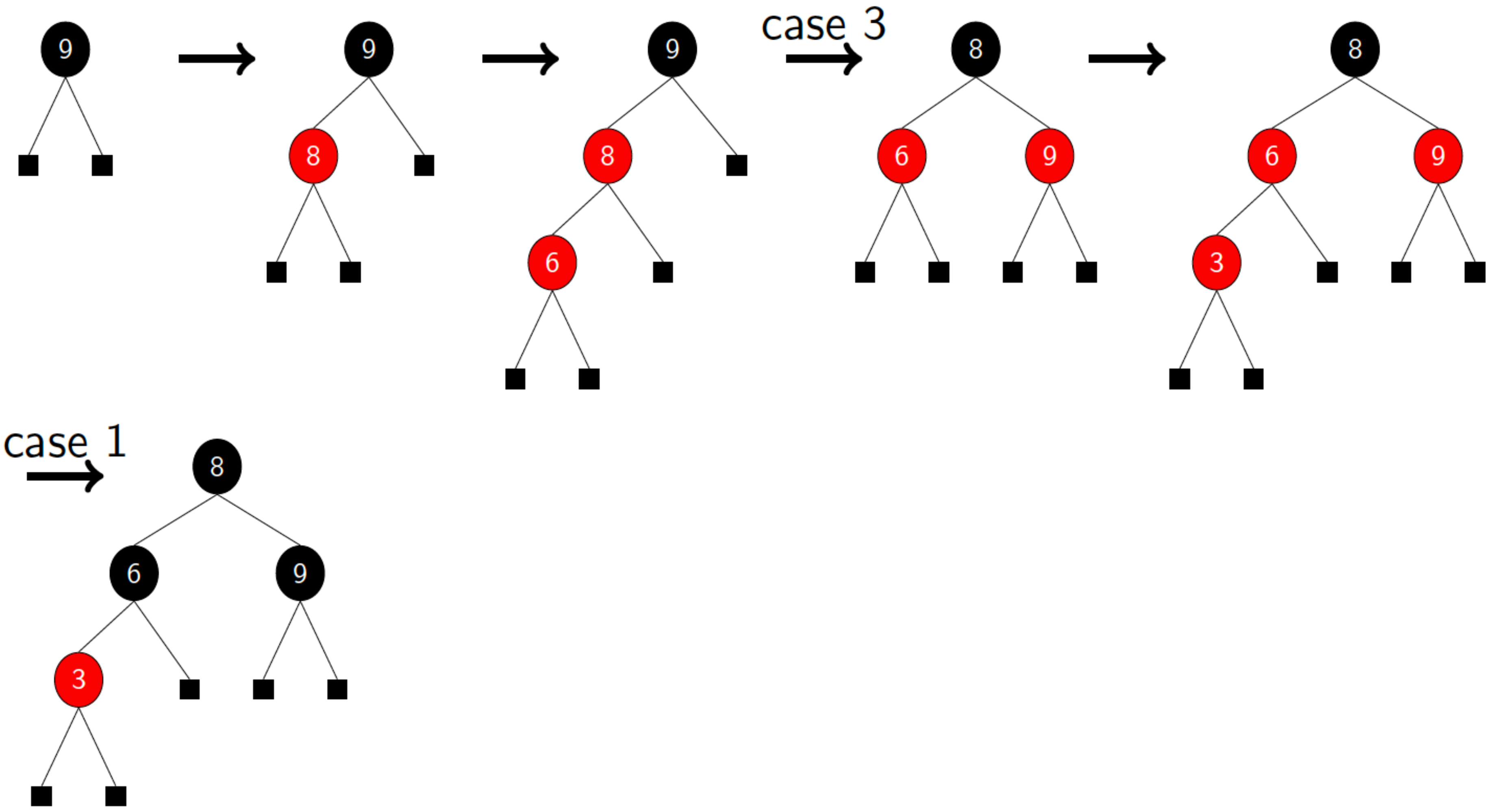
Example Insertions (9, 8, 6, 3, 5)



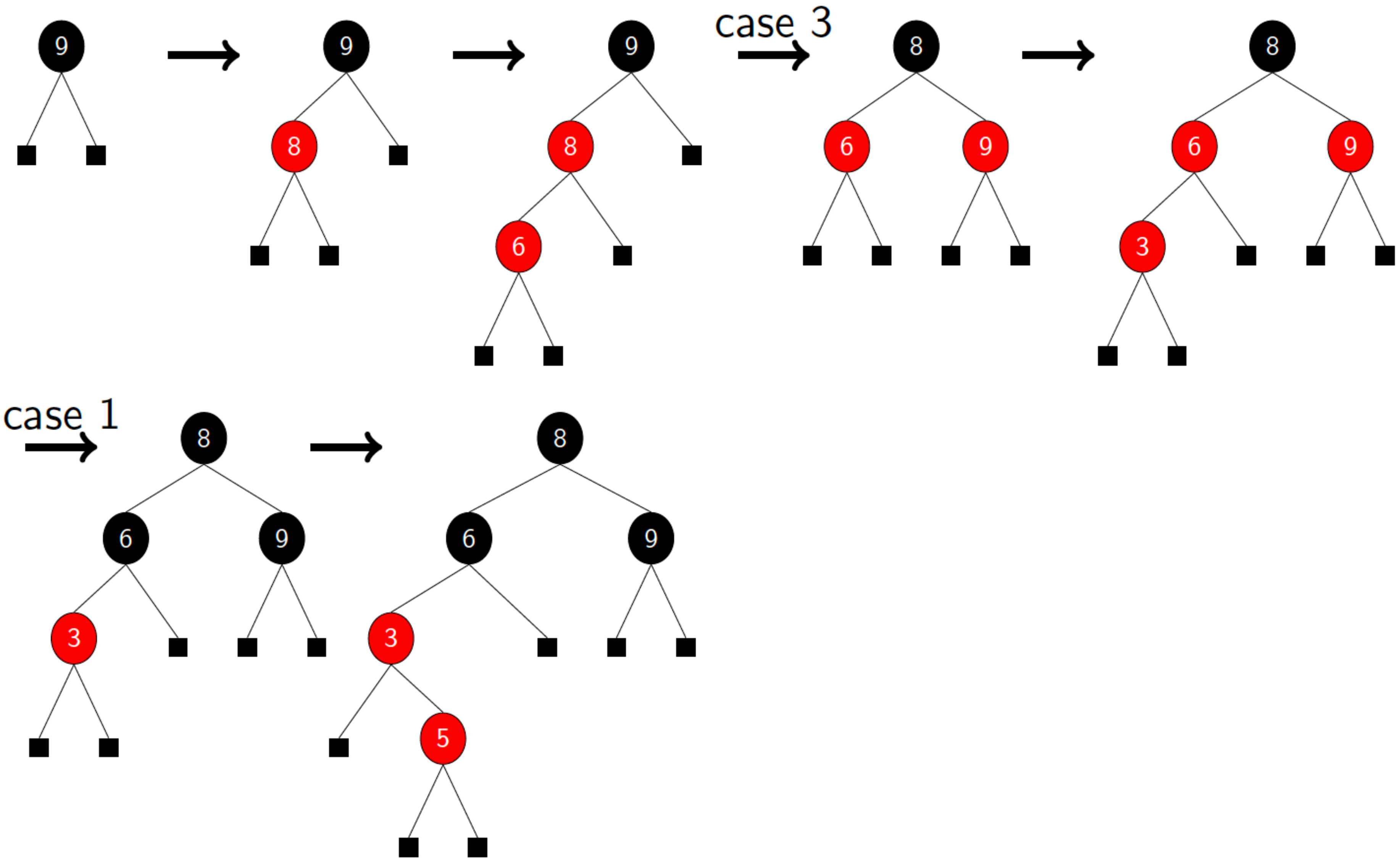
Example Insertions (9, 8, 6, 3, 5)



Example Insertions (9, 8, 6, 3, 5)



Example Insertions (9, 8, 6, 3, 5)



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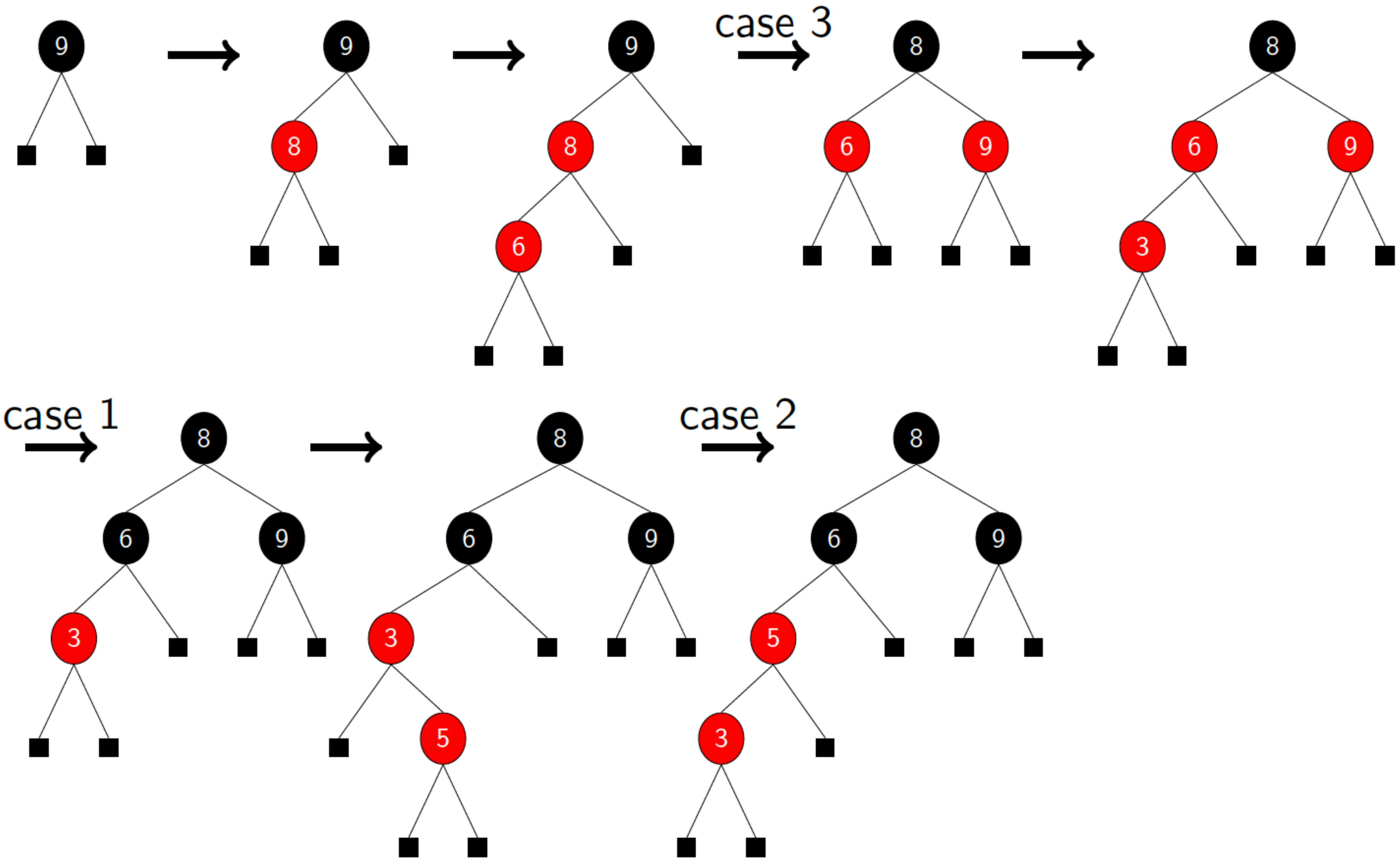
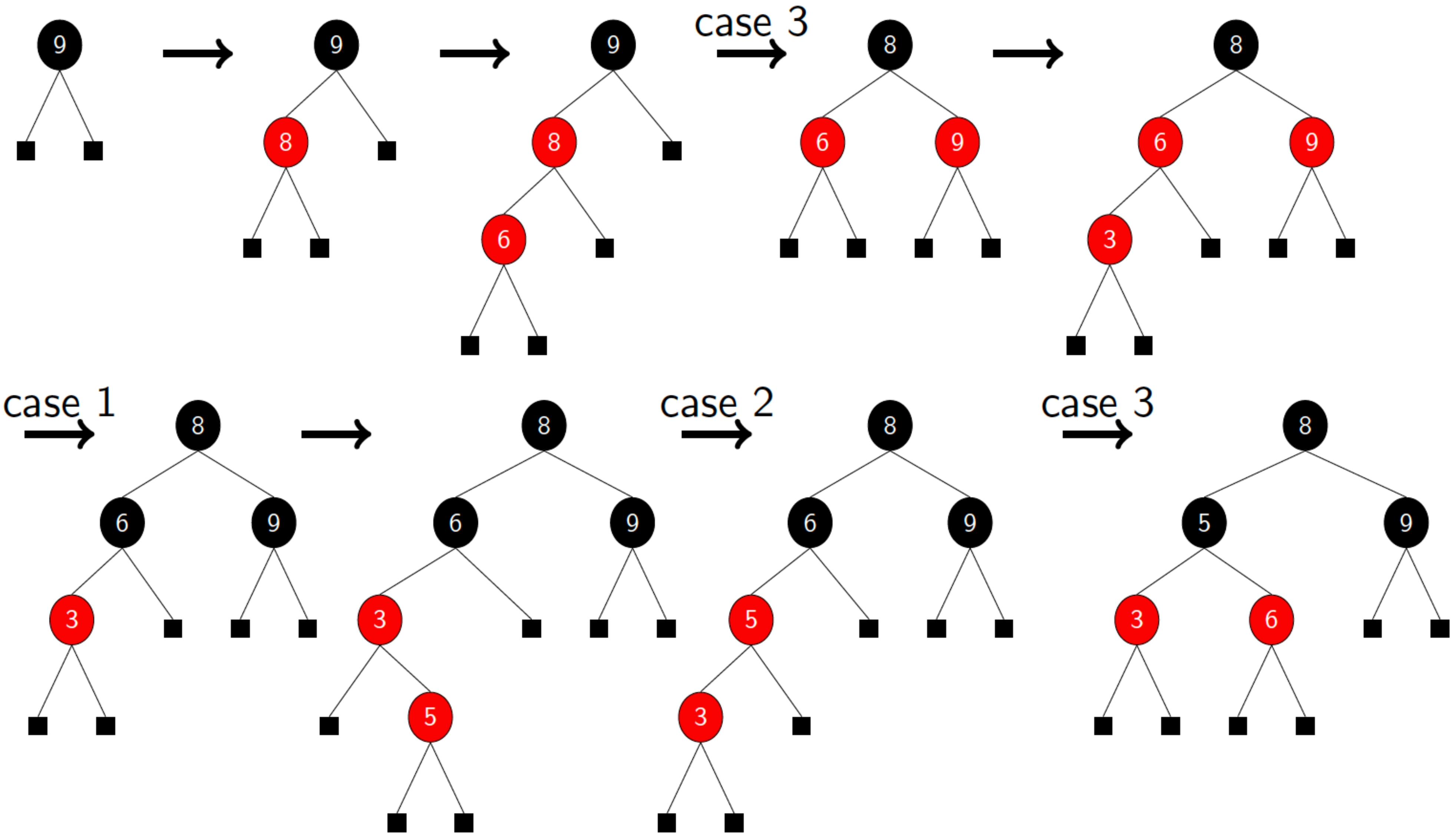


Example Insertions (9, 8, 6, 3, 5)



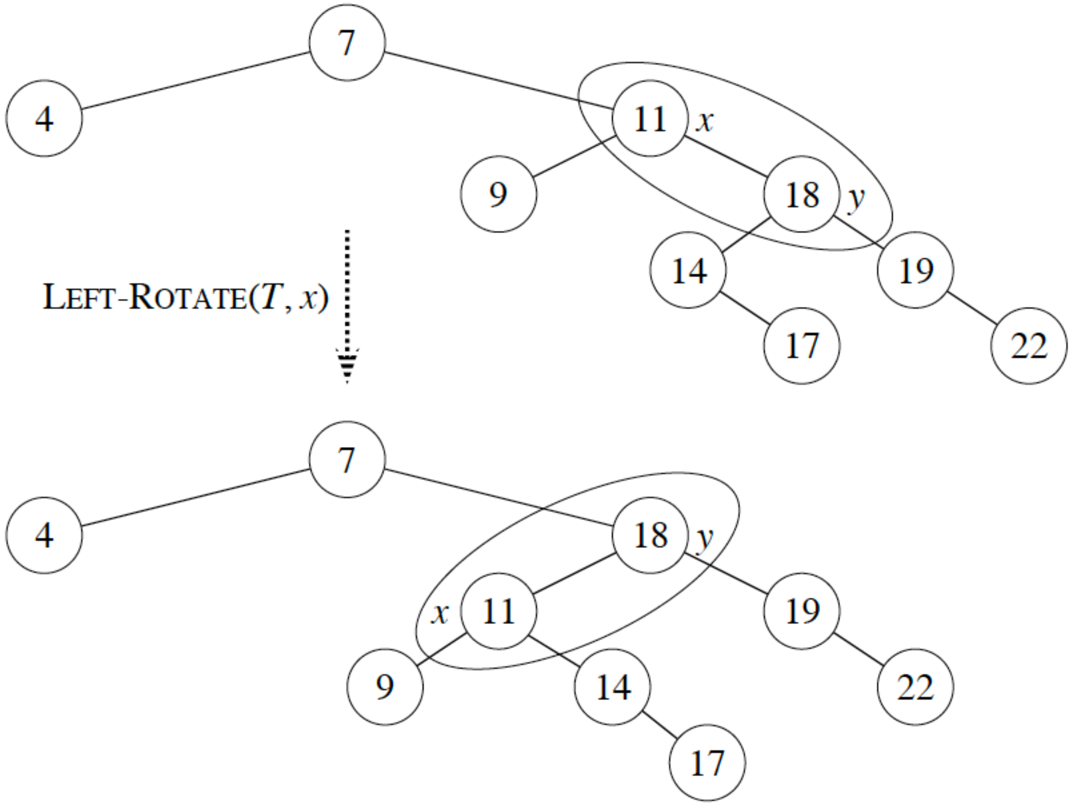
Suggested reading

Today's lecture covered sections 13.2 and 13.3.

Solutions

Class challenge

Perform an RBT_Rotate_left(T, x) operation:



Node colours omitted for convenience.