1 Tutorial problems

1. *Let* $X = \{1, 2, 5\}$ *and* $Y = \{0, 2, 4, 6\}$. *List the elements of:*

   (a) $X \cup Y$ *(the union of* X *and* Y*)*

   $$\{0, 1, 2, 4, 5, 6\}$$

   (b) $X \cap Y$ *(the intersection of* X *and* Y*)*

   $$\{2\}$$

   (c) $X - Y$ *(the complement of* Y *relative to* X*)*

   $$\{1, 5\}$$

   (d) $Y - X$ *(the complement of* X *relative to* Y*)*

   $$\{0, 4, 6\}$$

   (e) $\mathcal{P}(X)$ *(the power set of* X*, i.e. the set of all subsets of* X*)*

   $$\emptyset, \{1\}, \{2\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 5\}, \{1, 2, 5\}$$

2. *Let* $X = \{0, 1, 2\}$ *and* $Y = \{f, t\}$.

   (a) *List all the members of* $X \times Y$.

   $$(0, f), (0, t), (1, f), (1, t), (2, f), (2, t).$$

   (b) *List all total functions from* Y *to* X.
   Each column (after the first) in the table below describes one function according to its values at f and t respectively.

f	0	0	0	1	1	1	2	2	2
t	0	1	2	0	1	2	0	1	2

(c) *List all partial (but not total) functions from Y to X.*

Each column (after the first) in the table below describes one function according to its values at f and t respectively. I use $\perp$ to represent "undefined".

f	$\perp$	$\perp$	$\perp$	$\perp$	0	1	2
t	$\perp$	0	1	2	$\perp$	$\perp$	$\perp$

3. Let X be a set with 3 elements, and Y a set with 4 elements.

(a) *How many elements are there in $X \times Y$?*

12 since any one of the 3 elements from X could come first, and any of the 4 from Y second (as these are independent we multiply together the options).

(b) *How many total functions are there from X to Y?*

Each of the 3 elements of X independently chooses one of 4 values, so we get $4 \times 4 \times 4 = 4^3 = 64$ possible total functions.

(c) *How many total functions are there from Y to X?*

This time $3^4 = 81$

(d) *How many partial (possibly total) functions are there from X to Y?*

We can think of "undefined" as a fifth possible value so get $5^3 = 125$.

4. *Give examples of functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy:*

(a) *f is total and injective but not surjective,*

Note in this and all the remaining answers there are many possibilities. I'm just giving one example of each. The function $f(x) = 2x$ is injective since $2x = 2y$ implies $x = y$ and not surjective since it takes on no odd values.

(b) *f is total and surjective but not injective,*

Take $f(x) = x - 1$ except $f(0) = 0$. It's surjective since $x = f(x + 1)$ for all x but not injective since $f(0) = f(1)$. Or take $f(x) = \lfloor x/2 \rfloor$ (i.e., integer division as in Java). Now surjective because $x = f(2x)$ for all x, but not injective since $f(2k) = f(2k + 1)$ for any k.

(c) *f is total, injective and surjective, but is not the identity,*

Take $f(0) = 1$, $f(1) = 0$ and $f(x) = x$ for all $x > 1$.

(d) *f is not total, but is surjective.*
Take f to be undefined at odd numbers and $f(x) = x/2$ for even numbers, or f undefined at 0 and $f(x) = x - 1$ for all $x > 0$.