1 Equivalence relations

An *equivalence relation* is a particularly important type of relation on a set (i.e. from a set A to itself). Intuitively, equivalence relations identify subsets of A sharing some common feature. Equivalence relations behave very much like equality (and indeed the `equals` method in many programming languages generally implements some sort of equivalence relation), and so we tend to denote them by infix symbols such as $\sim$ (i.e. we write $a \sim b$ rather than $(a, b) \in E$).

The defining characteristics of an equivalence relation $\sim$ on a set A are:

Reflexivity For every $a \in A$, $a \sim a$.

Symmetry For every $a, b \in A$ if $a \sim b$ then $b \sim a$.

Transitivity For every $a, b, c \in A$ if $a \sim b$ and $b \sim c$ then $a \sim c$.

In potatoes and arrows terms this means that every element has an arrow from itself to itself, every arrow points both ways, and that chains of arrows are also represented by single arrows. That means that, for any $a \in A$, the set of those $b \in A$ such that $a \sim b$ forms a big clump of elements all related to one another. This clump is called the *equivalence class* of a and denoted $[a]$.

Theorem 1.1. *Let $\sim$ be an equivalence relation on A. The equivalence classes of $\sim$ form a partition of A. Specifically:*

- *for every $a \in A$, $a \in [a]$*
- *for every $a, b \in A$ either $[a] = [b]$ (if $a \sim b$) or $[a] \cap [b] = \emptyset$ (if $a \not\sim b$).*

More generally, there is a direct correspondence between equivalence relations and partitions. The direction from a relation to a partition is given above. But, if we have a partition we can define a relation by "$a \sim b$ if a and b lie in the same set of the partition" and it is easily seen that this is an equivalence relation whose equivalence classes are the sets of the partition.

2 How big is a set?

What do we mean when we say “NZ has five denominations of coins”, “I have five fingers”, “there are five weekdays”? What common property of these sets is captured by the number five?

10	thumb	Monday
20	index	Tuesday
50	middle	Wednesday
100	ring	Thursday
200	pinkie	Friday

What we see is that there is a one to one correspondence between any two of these sets, and indeed between any one of them and the set $\{1, 2, 3, 4, 5\}$ (or to be more CS-y, $\{0, 1, 2, 3, 4\}$).

That this was the correct definition of the “size” of a set was recognized by Cantor.

3 Cardinality

Definition 3.1. Two sets, X and Y, are the same size (equinumerous, have the same cardinality) if there is a one to one correspondence between them. In this case we write $card(X) = card(Y)$

Definition 3.2. We say that $card(X) \leq card(Y)$ if there is an injection from X to Y.

You can define all you want but there's a certain obligation to check that it all make sense. Among other things we should check that “having the same cardinality” behaves like an equivalence relation between sets, and that $card(X) \leq card(Y)$ and $card(Y) \leq card(X)$ implies that $card(X) = card(Y)$. The first of these is straight forward definition chasing, while the second is somewhat more delicate and is called the *Schröder-Bernstein theorem*.

4 Denumerable sets

A set is *finite* if we can (in principle) count its elements, that is X is finite if

$$card(X) = card(\{1, 2, \ldots, n\})$$

for some $n \in \mathbb{N}$. In that case we use a short form $card(X) = n$.

A set is *infinite* if it is not finite (d'uh)

In fact, a set is infinite if $card(\mathbb{N}) \leq card(X)$. That's not so easy to prove from the definitions. What this means is that there are no "small" infinite sets using Cantor's definitions, e.g. the set of powers of 10 has just the same cardinality as $\mathbb{N}$ itself (the map $i \mapsto 10^i$ is a bijection).

A set is *countable* or *countably infinite* if it is equinumerous with $\mathbb{N}$, and a set is *denumerable* if it is finite or countably infinite.

Some apparently large infinite sets turn out to be countable. For instance $\mathbb{Z}$ the set of all integers. We can define a bijection $f : \mathbb{N} \to \mathbb{Z}$ by:

$$f(n) = \begin{cases} -n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

Even $\mathbb{N} \times \mathbb{N}$ is equinumerous with $\mathbb{N}$. We can begin to see how to construct a bijection by listing the elements of $\mathbb{N} \times \mathbb{N}$ in some sensible order that makes sure we eventually see each one:

0	(0,0)
1	(0,1)
2	(1,0)
3	(0,2)
4	(1,1)
5	(2,0)
6	(0,3)
…	

Writing a formula for this function is not so easy, but in the other direction (from $\mathbb{N} \times \mathbb{N}$

to $\mathbb{N}$) the following defines a bijection:

$$f(x, y) = \frac{(x + y)(x + y + 1)}{2} + x$$

But it's easier to prove the result by noting that it's obvious that $card(\mathbb{N}) \leq card(\mathbb{N} \times \mathbb{N})$ (e.g. $n \mapsto (n, 0)$), while the map $(x, y) \mapsto 2^x 3^y$ gives an injection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ showing that $card(\mathbb{N} \times \mathbb{N}) \leq card(\mathbb{N})$.

5 Uncountable sets

So, is there really only a two way distinction between finite and infinite? No, Cantor's main contribution in this area was to demonstrate that there are different sizes of infinity (in fact lots of them, but we won't worry about that). For our purposes the main reason for considering this result is because of a proof technique which it introduces, the *diagonal argument*, which we will find useful later in showing that certain problems are uncomputable.

Theorem 5.1. *The set* $\mathbb{N}^{\mathbb{N}}$ *of all functions from* $\mathbb{N}$ *to* $\mathbb{N}$ *is not denumerable.*

Proof. The proof of this result is "by contradiction". That means, we suppose it to be false and show that this leads to a contradiction. The only way to avoid the contradiction is for the result to be true, so there we are. Ok, so suppose that the result is false meaning that $\mathbb{N}^{\mathbb{N}}$ is denumerable. In that case there must exist a bijective function $F : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$. Since $F(n)$ is itself a function, let's denote it by f_n for each n just to keep things simple. What we are going to do is to construct a function $g : \mathbb{N} \to \mathbb{N}$ which is different from *all* the functions f_n. This g then is not the image of any element of $\mathbb{N}$ under F which will contradict the assumption that F is a bijection (and hence that $\mathbb{N}^{\mathbb{N}}$ is denumerable). The idea is to imagine all the functions f_n displayed in a giant table:

	0	1	2	3	4	...
f_0	$\underline{5}$	2	6	1	3	...
f_1	3	$\underline{1}$	4	1	5	...
f_2	0	0	$\underline{0}$	0	0	...
f_3	0	1	4	$\underline{9}$	16	...
$\vdots$						$\ddots$

In the table we have highlighted the diagonal elements – in other words the values $f_k(k)$. The idea to ensure that $g \neq f_k$ will be to define it in such a way that it disagrees with $f_k(k)$. In particular we could set:

$$g(k) = f_k(k) + 1$$

for all k. So, it is not the case that $g = f_n$ for any n (since g and f_n disagree in at least one place) and we have the contradiction we wanted. □

6 Tutorial problems

1. Define a binary relation $\sim$ on $\mathbb{N}$ by: $n \sim m$ if and only if $n - m$ is a multiple of 12. Show that $\sim$ is an equivalence relation. How many equivalence classes does it have?

2. Define a binary relation $\sim$ on $\mathbb{N}$ by: $n \sim m$ if and only if n and m have the same first digit (written in base 10). Show that $\sim$ is an equivalence relation. How many equivalence classes does it have?

3. Show that the set of even natural numbers is denumerable.

4. Show that the set of even integers is denumerable.

5. Show that the set of total functions from $\mathbb{N}$ to $\{0, 1\}$ is uncountable.

6. A total function $f : \mathbb{N} \to \mathbb{N}$ is called *monotone increasing* if $f(n) < f(n+1)$ for all $n \in \mathbb{N}$. Prove that the set of monotone increasing functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.

7. (Harder) Show that, for any set A, $card(A) < card(\mathcal{P}(A))$.

8. (Harder) A function $f : \mathbb{N} \to \{0, 1\}$ is said to have *finite support* if it is non-zero only finitely many times (or, put another way, there is some n such that for all $m \geq n$, $f(m) = 0$). Show that the set of functions from $\mathbb{N}$ to $\{0, 1\}$ that have finite support is denumerable.