1 Tutorial problems

1. *Define a binary relation $\sim$ on $\mathbb{N}$ by: $n \sim m$ if and only if $n - m$ is a multiple of 12. Show that $\sim$ is an equivalence relation. How many equivalence classes does it have?*

   We need to show that the three properties of an equivalence relation are satisfied:

   Reflexive For any $n \in \mathbb{N}$, $n - n = 0 = 12 \times 0$, so is a multiple of 12. Thus $n \sim n$.

   Symmetric For any $n, m \in \mathbb{N}$, if $n \sim m$ then $n - m = 12k$ for some integer k, but then $m - n = 12(-k)$ is also a multiple of 12. Thus $m \sim n$.

   Transitive For any $a, b, c \in \mathbb{N}$ suppose that $a \sim b$ and $b \sim c$. So $a - b = 12k$ for some k and $b - c = 12j$ for some j. But then

   $$a - c = (a - b) + (b - c) = 12j + 12k = 12(j + k)$$

   is a multiple of 12. Thus $a \sim c$.

   The relation has 12 equivalence classes since any number is equivalent to its remainder on division by 12, and no two distinct numbers between 0 and 11 (inclusive) are equivalent.

2. *Define a binary relation $\sim$ on $\mathbb{N}$ by: $n \sim m$ if and only if n and m have the same first digit (written in base 10). Show that $\sim$ is an equivalence relation. How many equivalence classes does it have?*

   The word "same" in the definition makes verifying the three conditions required essentially trivial! There are 10 equivalence classes, one for each possible first digit (the first digit 0 occurs in the number 0 alone).

3. *Show that the set of even natural numbers is denumerable.*

   The map f defined by $f(x) = 2x$ is a bijection from the set of natural numbers to the set of even natural numbers.

4. *Show that the set of even integers is denumerable.*

   Define $f(x) = x$ for even natural numbers x and $f(x) = -x - 1$ for odd natural numbers x. The f is a bijection from the natural numbers to the even integers.

5. *Show that the set of total functions from* $\mathbb{N}$ *to* $\{0, 1\}$ *is uncountable.*

   Suppose for the sake of contradiction that it wasn't and therefore we had a bijection F from the natural numbers to this set. Let $f_i = F(i)$. Define the function $g : \mathbb{N} \to \{0, 1\}$ by:
   $$g(n) = 1 - f_n(n)$$
   Then $g(n) \neq f_n(n)$ and so the function g is not equal to the function f_n. But this is true for all n, contradicting the fact that F was supposed to be onto the set of functions from $\mathbb{N}$ to $\{0, 1\}$.

6. *A total function* $f : \mathbb{N} \to \mathbb{N}$ *is called* monotone increasing *if* $f(n) < f(n + 1)$ *for all* $n \in \mathbb{N}$. *Prove that the set of monotone increasing functions from* $\mathbb{N}$ *to* $\mathbb{N}$ *is uncountable.*

   As usual, work by contradiction. Suppose that we had a bijection F from $\mathbb{N}$ to the set of monotone increasing functions. Let $f_n = F(n)$. Define $g : \mathbb{N} \to \mathbb{N}$ by:
   $$g(n) = 1 + \max\{g(n - 1), f_n(n)\}$$
   The first part of the maximum guarantees that g is monotone increasing and the second part that $g \neq f_n$. Now we have the usual contradiction.

7. *(Harder) Show that, for any set* A, $card(A) < card(\mathcal{P}(A))$.

   This still falls under the standard approach. It's a bit easier to understand if we first note that $\mathcal{P}(A))$ and the set of functions from A to $\{0, 1\}$ are equinumerous (to each subset associate the function that is 1 for elements of the subset, and 0 elsewhere). It's clear that $card(A) \leq card(\mathcal{P}(A))$, since the map that sends each element to the singleton set containing it is an injection. So to finish we only need to show that there's no bijective map between A and the set of functions from A to $\{0, 1\}$. But if F were such a bijection, define $f_a = F(a)$. Now define $g : A \to \{0, 1\}$ by
   $$g(a) = 1 - f_a(a)$$
   for each $a \in A$. Then $g \neq f_a$ since they disagree at a, and we have the usual contradiction.

8. *(Harder) A function $f : \mathbb{N} \to \{0, 1\}$ is said to have* finite support *if it is non-zero only finitely many times (or, put another way, there is some n such that for all $m \geq n$, $f(m) = 0$). Show that the set of functions from $\mathbb{N}$ to $\{0, 1\}$ that have finite support is denumerable.*

   It's enough to find a surjective map from $\mathbb{N}$ to the set of such functions. Given such a function write down its values in order finishing at the last 1. Imagine reading this number backwards as a binary number. Take the map which sends the natural number you read in this way to that function. Now that doesn't make sense as it stands since it's circular. Here's how to define it properly. For each positive integer n write

$$n = \sum_{i=0}^{\infty} a_i 2^i$$

   where $a_i \in \{0, 1\}$. There is a unique way to do this. Define the image of n to be the function whose value at i is a_i. This is surjective since, given a function of finite support we can "read its n" and it is then the image of that n.