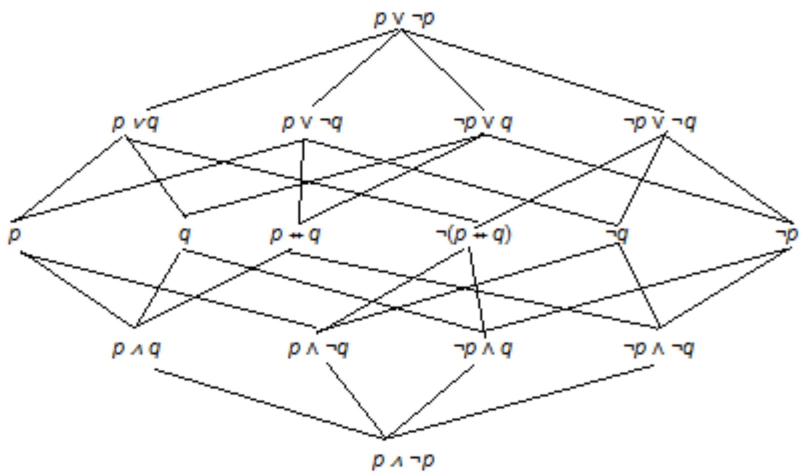
COSC410 Solutions to Assignment 1

1. Assume we have a language L_A with $A = \{p, q\}$ and $S = W_A = \{11, 10, 01, 00\}$.
   Give the Lindenbaum-Tarski algebra of propositions, using one representative sentence for each proposition.

Solution:



2. Show that $\varphi \equiv \psi$ iff every state s in S satisfies the sentence $\varphi \leftrightarrow \psi$.

Solution:
Suppose $\varphi \equiv \psi$.
Then Mod(φ) = Mod(ψ), i.e. the two sets have exactly the same elements.
Now we must show that all states satisfy the string $\varphi \leftrightarrow \psi$.
Let $s \in S$ be arbitrary.
There are exactly two cases: either s satisfies $\varphi \leftrightarrow \psi$ or it doesn't.
Suppose s doesn't satisfy $\varphi \leftrightarrow \psi$.
Then s does not give the same truth value to both φ and ψ (because, by the definition of satisfaction, if s gave the same truth value to φ and ψ, then s would satisfy $\varphi \leftrightarrow \psi$).
So either s satisfies φ but not ψ, or s satisfies ψ but not φ,
i.e. either $s \in$ Mod(φ) but $s \notin$ Mod(ψ), or $s \notin$ Mod(φ) but $s \in$ Mod(ψ).
Thus Mod(φ) and Mod(ψ) do not have exactly the same members.
But this contradicts the fact that Mod(φ) = Mod(ψ).
We therefore conclude that s satisfies $\varphi \leftrightarrow \psi$.
And since s was chosen arbitrarily, every state satisfies $\varphi \leftrightarrow \psi$.

Conversely, suppose that every state s in S satisfies the sentence $\varphi \leftrightarrow \psi$.
Now we must show that $\varphi \equiv \psi$, i.e. that Mod(φ) = Mod(ψ).
There are exactly two cases: either Mod(φ) = Mod(ψ) or not.

If not, then there is some state s that belongs to one of the sets but not to the other.
Suppose $s \in \text{Mod}(\varphi)$ but $s \notin \text{Mod}(\psi)$.
Then s fails to satisfy $\varphi \leftrightarrow \psi$, contradicting our starting assumption.
On the other hand, suppose $s \notin \text{Mod}(\varphi)$ but $s \in \text{Mod}(\psi)$.
Then again s fails to satisfy $\varphi \leftrightarrow \psi$, contradicting our starting assumption.
We conclude that $\text{Mod}(\varphi) = \text{Mod}(\psi)$, i.e. $\varphi \equiv \psi$.

3. We show that $\models$ is contrapositive, i.e. that if $\alpha \models \beta$ then $\neg\beta \models \neg\alpha$.

Solution:
Suppose $\alpha \models \beta$, i.e. that every model of α is also a model of β.
Now we must show that $\neg\beta \models \neg\alpha$.
Pick any model of $\neg\beta$, and call it s.
There are exactly two cases: either s satisfies α or not.
We want to show that s satisfies $\neg\alpha$, i.e. that s does not satisfy α.
So imagine that s does satisfy α.
SInce $\alpha \models \beta$, s will also satisfy β.
But we picked s to be a model of $\neg\beta$, so s cannot satisfy β.
Hence we have a contradiction.
So we may eliminate the possibility that s satisfies α.

4. Suppose $A = \{p_0, p_1, p_2\}$. Consider the sentence $p_1 \leftrightarrow p_2$. Give two equivalent sentences, one in SDNF and the other in CNF (conjunctive normal form).

Solution:
First we need to find the models of $p_1 \leftrightarrow p_2$. Just go through S = {111, 110, 101, 011, 100, 010 001, 000} and see which of these give the same truth value to both p_1 and p_2.
We see that $\text{Mod}(p_1 \leftrightarrow p_2)$ = {111, 011, 100, 000}.

Now take the state descriptions of the models and hook them together with disjunctions to get the SDNF:
$(p_0 \wedge p_1 \wedge p_2) \vee (\neg p_0 \wedge p_1 \wedge p_2) \vee (p_0 \wedge \neg p_1 \wedge \neg p_2) \vee (\neg p_0 \wedge \neg p_1 \wedge \neg p_2)$.

For CNF we can use the nonmodels of $p_1 \leftrightarrow p_2$, and since we know what the models of this sentence are it is easy to find the nonmodels: 110, 101, 010 and 001.

We want to exclude each of 110, 101, 010 and 001, and a sentence that does this is:

$$\neg(p_0 \wedge p_1 \wedge \neg p_2) \wedge \neg(p_0 \wedge \neg p_1 \wedge p_2) \wedge \neg(\neg p_0 \wedge p_1 \wedge \neg p_2) \wedge \neg(\neg p_0 \wedge \neg p_1 \wedge p_2).$$

Now use De Morgan's law: $\neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta)$.
The above sentence simplifies to the CNF:

$$(\neg p_0 \vee \neg p_1 \vee p_2) \wedge (\neg p_0 \vee p_1 \vee \neg p_2) \wedge (p_0 \vee \neg p_1 \vee p_2) \wedge (p_0 \vee p_1 \vee \neg p_2).$$