COSC 410 Lecture 8
Temporal Logic

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2016

Introduction

Suppose we want to enrich the basic propositional language L_A by including temporal adverbs so as to express safety and liveness properties like "The system will **always** run (i.e. will never crash in future)" and "If a request is made to print a file, **eventually** the file will be printed". Or if you don't like operating systems, think of expressing ideas such as "In future I will always write to thank Aunt Maud for her present " and "Ever since I forgot to thank Aunt Maud for her present she has been threatening to change her will".

Temporal logics allow us to express such ideas. We look at two different temporal logics, one simpler but weaker than the other.

Prior's logic of Future and Past

A simple plan is to extend L_A by adding two modal box operators $[F]$ and $[P]$, allowing us to talk broadly about the future and the past. A sentence of the form $[F]\varphi$ will be interpreted as saying "At all *future* times φ is *going* to be the case", or more briefly "Henceforth φ is the case". Similarly $[P]\varphi$ will be interpreted as "At all *past* times φ *has* been the case", or more briefly "Hitherto φ has been the case".

To express the idea that something will happen eventually, i.e. at some future time, diamond operators are useful, and so we take $\langle F\rangle$ as an abbreviation for $\neg[F]\neg$ and $\langle P\rangle$ as an abbreviation for $\neg[P]\neg$.

The new language $L_A^{\{F,P\}}$ will be equipped with a semantics producing the *tense logic* invented by the New Zealander Arthur Prior in 1957 to express the tenses in natural language (future and past).

Note that in the literature $[F]$ is often replaced by G (for "going to be the case" and $[P]$ is often replaced by H (for "Hitherto"). Our choice of $[F]$ and $[P]$ follows the convention of Goldblatt's excellent book, *Logics of Time and Computation.* Rob Goldblatt is a New Zealand logician at Victoria University of Wellington.

Kamp's logic of Since and Until

A more powerful temporal logic can be obtained by adding two binary modal operators $\mathcal{S}$ ("since") and $\mathcal{U}$ ("until") to the basic language L_A.

The operator "until" is important in the study of operating systems and concurrency, because one is typically interested in checking properties such as "The system will not respond **until** a request has been received". Of course, it is important in everyday life too, as in "Aunt Maud won't renew my allowance until I apologise to the vicar for getting drunk and singing Stairway to Heaven in the graveyard".

The language built with $\mathcal{S}$ and $\mathcal{U}$ is more expressive than Prior's tense logic because we can obtain the operators $[F]$ and $[P]$ as abbreviations for expressions involving $\mathcal{S}$ and $\mathcal{U}$, whereas we cannot get $\mathcal{S}$ and $\mathcal{U}$ as abbreviations for expressions built up from $[F]$ and $[P]$.

The binary connectives $\mathcal{S}$ and $\mathcal{U}$ were invented by Hans Kamp in his PhD thesis at UCLA in 1968.

A Pause to Contemplate the Nature of Time

The semantics of a temporal logic is based on our conceptual model of time. The most common approach to modelling time is to let our set S of states be a set of time instants. The usual choice is to take one of the number sets as our set of time instants. We might use

- the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$, with the usual ordering $n < m$ telling us that instant n occurs earlier than m, if we want time to be
  - discrete in the sense that for every instant there is a next instant (for example, time point n has its successor $n + 1$ as the next instant)
  - bounded towards the past, i.e. to have a starting point;
- the set of integers $\mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$ if we want time to be discrete but unbounded in both directions (past and future);
- the set of rational numbers $\mathbb{Q} = \{\frac{n}{m} \mid n, m \in \mathbb{Z} \text{ and } m \neq 0\}$ if we want time to be dense instead of discrete, where by density we mean that between any two time points x and y with $x < y$ there is another time point z such that $x < z$ and $z < y$;
- the set of real numbers $\mathbb{R}$ if we want time not only to be dense but also continuous, i.e. having no gaps (mathematically, every nonempty subset with an upper bound must have a least upper bound[1]).

For applications in computing and artificial intelligence, discrete time is usually preferred, and so we will take $S = \mathbb{Z}$ for the rest of this lecture.

Notice that using $\mathbb{Z}$ with its ordering $<$ not only gives us a next instant for every time point, it also makes time *linear*: we can visualise the set $\mathbb{Z}$ of time points as a line stretching infinitely far towards the left and towards the right. Mathematically, the property of linearity is the following: if n and m are any points in S then $n < m$ or $n = m$ or $m < n$. In other words, if you take any

[1] In $\mathbb{Q}$ the set $\{x \mid x < \sqrt{2}\}$ has upper bounds (e.g. 2) but no least upper bound, because $\sqrt{2}$ is not rational. In this sense, $\mathbb{Q}$ has a gap that $\mathbb{R}$ does not have.

two different time points m and n, then $<$ forces them into line with one of them sitting to the left of the other.

Be aware that for some applications one might not want linearity but instead might prefer branching time, for instance if nondeterminism means that a given instant could have two or more different successors (or "next" instants). The temporal structure might branch only in a forward direction, considering the past as fixed, or it might be even more complicated and branch backward too. We will look at branching time briefly in the next lecture (on BDI logics).

You should also be aware that it is possible to use a conceptual model of time based on intervals instead of time points, where the intuition is that all we can really measure are intervals, not exact time instants. The original paper giving the interval-based approach is JF Allen (1983): Maintaining knowledge about temporal intervals, *Communications of the ACM* 26(11):832–843. We won't explore this option.

Semantics of Prior's Tense Logic $L_A^{\{F,P\}}$

A semantics for $L_A^{\{F,P\}}$ must have three parts:

- a set S of time instants (we will take $S = \mathbb{Z}$, the set of integers)
- a labelling function $V : S \longrightarrow W_A$ recording which atomic sentences are true at which instant (think of V as a sort of history or path of the system)
- an accessibility relation for each of $[F]$ and $[P]$, which we can take to be the usual ordering on the integers, $<$, and its converse $>$ respectively.

Since $>$ is the converse of $<$, i.e. just has mirror images of the ordered pairs of $<$, we can stick to using $<$ and think of accessibility going in different directions along this ordering — from left to right for $[F]$ and from right to left for $[P]$.

Expressing information about the future and the past

We want to say what it means for a state to satisfy a sentence. Our states will be time instants. Through the labelling function, we also have the idea of various elementary facts being true in a given state.

Definition 1 *(Satisfaction of* $[F], [P], \langle F\rangle, \langle P\rangle$*) At time n the sentence*

- $[F]\varphi$ *is satisfied iff* φ *is satisfied at every time* m *such that* $n < m$
- $[P]\varphi$ *is satisfied iff* φ *is satisfied at every time* k *such that* $k < n$
- $\langle F\rangle\varphi$ *is satisfied iff* φ *is satisfied at some time* m *such that* $n < m$
- $\langle P\rangle\varphi$ *is satisfied iff* φ *is satisfied at some time* k *such that* $k < n$.

So $[F]\varphi$ is true at time n if φ is true at all times in the future of n, whereas $[P]\varphi$ is true at n if φ is true at all times in the past of n. Since we are using the ordering $<$, we can have $[F]\varphi$ true at n without having φ true at n, and similarly for $[P]\varphi$. (Of course, there is nothing to stop us from using the ordering $\leq$ if we want the future and the past both to include the present instant n.)

Example 1 *Suppose we observe, over an extended period, a particular traffic light and the behaviour of cars approaching it. (So this is the familiar Traffic System, except that we replace the perspective of a driver waiting briefly at the intersection by that of a long-term observer sitting comfortably in the coffee shop overlooking the intersection.)*

Let the atomic sentences be p and q where p stands for "The light is red" and q for "The oncoming car stops". So $W_A = \{11, 10, 01, 00\}$ as before, but now $S = \mathbb{Z}$ with accessibility relation $<$. The fact that $S \neq W_A$ means that the labelling function V becomes very important. There is no specific labelling function V that is obviously correct — it all depends on what happens, i.e. depends on the historical path followed by the system.

Suppose we want to model a path in which events unfold in a law-abiding fashion: the traffic light toggles between red and green as the clock ticks while the oncoming cars sensibly stop at red lights and drive through green lights.

In this case we could model the system by using $V : S \longrightarrow W_A$ given by

$$\begin{aligned} V(n) &= 00 \textit{ if } n \textit{ is even} \\ V(n) &= 11 \textit{ if } n \textit{ is odd.} \end{aligned}$$

In this conceptual model:

- *at time $n = 1$, $p \wedge q$ is true since $V(1) = 11$ which satisfies p and satisfies q*
- *at time $n = 2$, $\neg p \wedge \neg q$ is true since $V(2) = 00$*

and so on.

We would expect at any given time n that $[F]p$ should not be satisfied because the light shouldn't stay red for all future time, but $\langle F\rangle p$ should be satisfied because the light should be red at some future times. And indeed:

- *n satisfies $\langle F\rangle p$ because if we take any odd m such that $n < m$ then $V(m) = 11$ so that m satisfies p and hence n satisfies $\langle F\rangle p$ (to see that we can always find such an odd m, note that either n is even or n is odd, and if n is even then $n + 1$ is odd while if n is odd then $n + 2$ is also odd)*
- *n fails to satisfy $[F]p$ because if we take any even m such that $n < m$ then $V(m) = 00$ so m fails to satisfy p and thus n fails to satisfy $[F]p$.*

Similarly, at any time n we have that $\langle P\rangle q$ is satisfied but $[P]q$ is not. To see that the former is the case, we need only note that whatever the value of n may be, there is some $k < n$ such that k is odd, and so $V(k) = 11$, which satisfies q. To see that n does not satisfy $[P]q$ we need only note that there are instants $k < n$ which are even, so that $V(k) = 00$.

It should be clear that the labelling function V plays a key role. If we change to $V' : S \longrightarrow W_A$ given by

$$\begin{aligned} V'(n) &= 00 \textit{ if } n \leq 5 \\ V'(n) &= 11 \textit{ if } n > 5 \end{aligned}$$

then at time $n = 5$ the sentence $[F]p$ is true because at every m such that $n < m$ we will have $V'(m) = 11$ so that p is true, and similarly $n = 5$ satisfies $[P]\neg p$. So this labelling function suits the situation in which the traffic light was green until, at time = 5, it switched to red, and broke, and stayed that way forever (and oncoming vehicles were all driverless cars from Google with *autonomous navigation systems programmed to stop at red lights and so they patiently waited forever for the light to change).*

The definition of satisfaction implements our desire to read $[F]$ as "henceforth", $[P]$ as "hitherto", $\langle F\rangle$ as "eventually", and $\langle P\rangle$ as "at some previous time". Let's think about other concepts we may wish to express.

Expressing information about Now, Always, and Sometimes

How might we say that *now*, at this very moment, φ is true? Well, that's easy. Whenever we might want to say this, the system will be at a time point, say n, and simply uttering φ at time n has precisely the same effect as saying that "Now, at this very moment when the clock says the time is n, φ is the case". In other words, we don't need a special symbol in the language for "now". Whether "Now, φ" is true at time n is just a matter of whether n satisfies φ.

How might we say that φ is *always* true (i.e. that φ was true at all times in the past, is true now, and will be true at all times in the future)? Well, whatever the current time n is, if n satisfies

$$[P]\varphi \wedge \varphi \wedge [F]\varphi$$

then

- n satisfies $[P]\varphi$ so that φ is true at all past times
- n satisfies φ so φ is true now, and
- m satisfies $[F]\varphi$ so that φ is true at all future times.

We may therefore introduce a new symbol $\Box$ into the language $L_A^{\{F,P\}}$ by taking $\Box\varphi$ to be an abbreviation of $[P]\varphi \wedge \varphi \wedge [F]\varphi$, and we may read $\Box$ as "always".

Similarly we may take $\Diamond\varphi$ to be an abbreviation for $\langle P\rangle\varphi \vee \varphi \vee \langle F\rangle\varphi$ and read $\Diamond$ as "sometimes". The symbols look just like those we encountered in epistemic logic but they are being used to express very different ideas in the temporal context.

Semantics of Kamp's Temporal Logic $L_A^{\{S,U\}}$

When one models a hardware device such as a flipflop, one wants to be able to express, for example, that the output values do not change until the clock is pulsed. To represent "until" we need to think of it as a binary connective relating two sentences φ and ψ, and saying that φ will be true at least until some future instant at which ψ is true. The obvious analog of this involving past time is "since", where we want to say that φ has been true ever since some point in the past when ψ was true. We get $L_A^{\{S,U\}}$ by adding to an ordinary propositional language L_A two binary connectives $\mathcal{U}$ and $\mathcal{S}$ which are applied to sentences to build bigger sentences in the same way as other binary connectives such as $\wedge$ or $\rightarrow$ would be.

We may still take the semantics of the language $L_A^{\{S,U\}}$ to be given by the set $S = \mathbb{Z}$ of time instants, equipped with the usual relation $<$ on $\mathbb{Z}$ to give accessibility into the future direction (left to right) for "until" and into the past direction (right to left) for "since". We also need a labelling function $V : \mathbb{Z} \longrightarrow W_A$ to connect time instants to atomic sentences and their truth values, just as before. The major change is to the definition of satisfaction.

Expressing information about Until and Since

Definition 2 *(Satisfaction of $\mathcal{U}$ and $\mathcal{S}$) At time n the sentence*

- *$\varphi\,\mathcal{U}\,\psi$ is satisfied iff there exists a time t such that $n < t$ and t satisfies ψ, and m satisfies φ for all m such that $n < m$ and $m < t$*
- *$\varphi\,\mathcal{S}\,\psi$ is satisfied iff there exists a time t such that $t < n$ and t satisfies ψ, and m satisfies φ for all m such that $t < m$ and $m < n$.*

So $\varphi\,\mathcal{U}\,\psi$ is true now (at time n) if there is a future point at which ψ is true, with φ true at all instants between now and then.

However, we must beware of making unguarded assumptions about satisfaction. The fact that $\varphi\,\mathcal{U}\,\psi$ is true now does not mean that φ is true now (maybe it is, maybe it isn't). The technical point is that we have chosen to use $<$, not $\leq$, and so our definition of satisfaction does not require φ to be true at the instant n corresponding to now.

Nor does it mean that ψ is false now (maybe it is, maybe it isn't). Our technical choice of $<$ means there has to be some future time t at which ψ is true, but does not rule out the possibility that ψ may be true now.

Nor does the definition mean that ψ is false between now and the future instant t (maybe, maybe not).

Nor does it mean that φ has to become false when ψ is true (maybe it does, maybe it doesn't). This is perhaps the most tempting pitfall. But consider modelling "Aunt Maud is withholding Cousin Aubrey's allowance until he apologises to the vicar for stealing the communion wine and throwing a Halloween party in the graveyard". It is conceivable that Aunt Maud is so shocked by Aubrey's licentious behaviour that she may continue to withhold his allowance even after he apologises to the offended vicar. Our definition of satisfaction allows this possibility, by not expressing itself about what happens to φ at future time t.

So if we're modelling something like a flipflop and want to ensure that φ becomes false when ψ becomes true, we'll need to say more than just that $\varphi\,\mathcal{U}\,\psi$.

Recovering the concepts of Future, Past, Always, Sometimes

One of the most important features of the language $L_A^{\{S,U\}}$ is that we can define in it all the temporal adverbs of interest to us so far.

How can we get back the simple future and past operators $[F]$ and $[P]$ in the language $L_A^{\{S,U\}}$?

The sneaky trick we use is one of the few occasions I've seen where a tautology is useful.

Choose your favourite tautology (say, $p \vee \neg p$ where p is an arbitrary member of A) and let's call it $\top$ (read: top).

Now notice that n satisfies $\top\,\mathcal{U}\,\varphi$ iff there exists a time t such that $n < t$ and t satisfies φ (and we know $\top$ is always true in between, so we can ignore that part). Thus we may take $\langle F\rangle\varphi$ as an abbreviation for $\top\,\mathcal{U}\,\varphi$. And once we have the diamond, we can define $[F]\varphi$ as the abbreviation for $\neg\langle F\rangle\neg\varphi$, so that $[F]\varphi$ is really $\neg(\top\,\mathcal{U}\,\neg\varphi)$.

Similarly, notice that n satisfies $\top \mathcal{S} \varphi$ iff there exists a time t such that $t < n$ and t satisfies φ, so that we may take $\langle P \rangle \varphi$ as an abbreviation for $\top \mathcal{S} \varphi$. Hence $[P]\varphi$ abbreviates $\neg\langle P \rangle \neg\varphi$ which is equivalent to $\neg(\top \mathcal{S} \neg\varphi)$.

Having recovered $[F]$ and $[P]$ we can use them to define $\Box$ ("always") and $\Diamond$ ("sometimes") as before.

The useful idea of Next

One of the reasons for deciding to use a set of discrete time points on an unbranching line like $\mathbb{Z}$ was that at any time n there is a unique *next* instant $n + 1$. That is, from any state n of the system, we can move to only one 'next' state. This makes sense if we are modelling an agent who observes the system and doesn't influence it by performing actions. (If we wanted to model an agent who could change the state in various ways by choosing different actions, then we would need to use a branching time model. We could still have the idea of a 'next' state on each branch, but there would not be a unique next state.)

We can define a "next" operator using $\mathcal{U}$.

Take $\bot$ (read: bottom) to be your favourite contradiction, say $p \wedge \neg p$.

Now $\bot \mathcal{U} \varphi$ is satisfied at n iff there is a time t such that $n < t$ and t satisfies φ, and $\bot$ is satisfied by every m such that $n < m$ and $m < t$. We know that $\bot$ can never be satisfied. So there cannot be any values of m between n and t. So if n satisfies $\bot \mathcal{U} \varphi$, it means that some future t satisfies φ and it also means that t has to be the next instant after n.

Introduce the new symbol $\bigcirc$ (read: next) by taking $\bigcirc\varphi$ to abbreviate $\bot \mathcal{U} \varphi$.

Metalogic

Let us explore temporal logic by trying to prove some elementary properties that exploit the semantics we have chosen.

Suppose φ is true now. Then surely it is going to be the case at every future time that φ was true at some point in the past.

Theorem 1 *Let $n \in S$ arbitrarily. If n satisfies φ then n satisfies $[F]\langle P \rangle \varphi$.*

Proof. *Suppose n satisfies φ.*

Pick any m such that $n < m$.

Then m satisfies $\langle P \rangle \varphi$ since $n < m$ and n satisfies φ.

Hence n satisfies $[F]\langle P \rangle \varphi$. ■

Corollary 1 *Every $n \in S$ satisfies $\varphi \to [F]\langle P \rangle \varphi$, for every sentence φ.*

The proof of the above theorem relies on the fact that we use the same accessibility relation $<$ for both future and past, so that if m is in the future of n then n is in the past of m.

Suppose we don't know whether φ is true now, but we do know it was true at some time in the past. Is it still the case at every future moment that φ was true at some point in the past?

Theorem 2 *If n satisfies $\langle P\rangle\varphi$ then n satisfies $[F]\langle P\rangle\varphi$.*

Proof. *Suppose n satisfies $\langle P\rangle\varphi$.*

Then there is some k such that $k < n$ and k satisfies φ.

Pick any m such that $n < m$.

Then m satisfies $\langle P\rangle\varphi$ since $k < n$ and $n < m$ so by transitivity $k < m$, and k satisfies φ.

Hence n satisfies $[F]\langle P\rangle\varphi$. ■

Corollary 2 *Every $n \in S$ satisfies $\langle P\rangle\varphi \rightarrow [F]\langle P\rangle\varphi$, for every sentence φ.*

The proof of the above theorem relies on the fact that our accessibility relation $<$ is transitive. If we ever choose to model time with an accessibility relation that is not transitive, expect funny things to happen, such as this obvious property no longer holding.

For what sort of concept of time might you use a non-transitive ordering of time instants? Well, suppose you think of time as a circle. Then you might have a notion of 'before', for which we could use the symbol $<$. And suppose time r is before time s and time s is before time t and time t is before r because of looping around the circle. Now if you want the comparison relation 'before' to be irreflexive, i.e. if you want to rule out the possibility of an instant being before itself, which makes sense, then your relation $<$ that implements your concept of 'before' cannot be transitive. (If it was transitive, you would be able to reason that $r < s$ and $s < t$, so $r < t$ by transitivity, and now $r < t$ and $t < r$ so that $r < r$ by transitivity again. This contradicts the irreflexivity you wanted, i.e. you now have that instant r is before itself.)

Moving on, recall that in the case of epistemic logic, the reflexivity of the equivalence relations in the semantics allowed us to prove that if a state w satisfied $[i]\varphi$ then w satisfied φ. In our temporal semantics, we are using the relation $<$ which is not reflexive, in fact $<$ is *irreflexive* in the sense that there is no $n \in S$ such that $n < n$. So we should not have an analogous property saying that if n satisfies $[F]\varphi$ then n satisfies φ. Let us build a counterexample.

Theorem 3 *It need not be the case that if n satisfies $[F]\varphi$ then n satisfies φ.*

Proof. *Take $A = \{p, q\}$ and let $V : \mathbb{Z} \longrightarrow \{11, 10, 01, 00\}$ be given by*

$$\begin{aligned} V(m) &= 11 \textit{ if } m \leq 5 \\ V(m) &= 00 \textit{ if } m > 5. \end{aligned}$$

Now take $n = 5$.

If m is such that $n < m$ then $V(m) = 00$ and m satisfies $\neg p$.

Thus n satisfies $[F]\neg p$.

But n fails to satisfy $\neg p$, for $V(n) = V(5) = 11$. ■

Let's check that "until" is well-behaved.

Theorem 4 *If n satisfies $\varphi\,\mathcal{U}\,\psi$ then n satisfies $\langle F\rangle\psi$.*

Proof. *Suppose n satisfies $\varphi\,\mathcal{U}\,\psi$.*

Then there is some t such that $n < t$ and t satisfies ψ.

Hence n satisfies $\langle F\rangle\psi$. ■

We have deliberately selected our semantics so that every time point has an immediate successor. Let us return to exploring the consequences of this design choice.

Theorem 5 *Every n satisfies $(\varphi \wedge [P]\varphi) \rightarrow \langle F\rangle[P]\varphi$.*

Proof. *Suppose n satisfies $\varphi \wedge [P]\varphi$.*

Then n satisfies φ and k satisfies φ for every $k < n$.

Thus $n+1$ satisfies $[P]\varphi$.

Hence n satisfies $\langle F\rangle[P]\varphi$. ■

Our temporal logic is linear, so not only does every instant have a next instant, but the next instant is unique (because we do not have our time line splitting into branches). So if a sentence φ is true at the next instant, then $\neg\varphi$ cannot be true at that next instant. (In branching time logics, this can be different because you can have two different next instants, each on a different branch of time, and in one φ may be true while in the other $\neg\varphi$ may be true.)

Theorem 6 *Every n satisfies $\bigcirc\varphi \leftrightarrow \neg\bigcirc\neg\varphi$.*

Proof. *Suppose n satisfies $\bigcirc\varphi$.*

Recall that $\bigcirc\varphi$ abbreviates $\bot\,\mathcal{U}\,\varphi$.

If n satisfies $\bot\,\mathcal{U}\,\varphi$ then there must be some t in the future of n such that t satisfies φ and such that the contradiction $\bot$ is true at all points between n and t.

Because $\bot$ is not true at any point, there cannot be any points between n and t, so that we must have $t = n+1$.

Thus $n+1$ satisfies φ.

Thus $n+1$ fails to satisfy $\neg\varphi$.

Thus n fails to satisfy $\bigcirc\neg\varphi$.

Thus n satisfies $\neg\bigcirc\neg\varphi$.

Conversely, if n satisfies $\neg\bigcirc\neg\varphi$ then n fails to satisfy $\bigcirc\neg\varphi$.

Thus $n+1$ fails to satisfy $\neg\varphi$.

Thus $n+1$ satisfies φ.

Hence n satisfies $\bot\,\mathcal{U}\,\varphi$, i.e. $\bigcirc\varphi$. ■

Exercises

Quiz: The quiz question at the start of lecture 9 will come from exercises 1, 2, and 3 below.

1. Show that if n satisfies $[P]\varphi$ then n satisfies $[P][P]\varphi$. (This result follows from the transitivity of $<$, and should remind you of positive introspection in epistemic logic.)
2. Show that n satisfies $\bigcirc\varphi \rightarrow \langle F\rangle\varphi$.
3. Show that n satisfies $\bigcirc\neg\varphi \rightarrow \neg\bigcirc\varphi$. (This result follows from the linearity of $<$, which tells us we don't have two next instants on different branches of time.)

4. Show that if n satisfies $\langle P\rangle\varphi$ then n satisfies $[P](\langle P\rangle\varphi \vee \varphi \vee \langle F\rangle\varphi)$. (This result follows from the linearity of $<$, so you would not expect it to hold in branching time logics, or at least not those that branch into the past. Suppose we want a property that holds for linear time logics but not in a logic based on time branching into the future. Can you think of what sentences might express such a property?)

5. Suppose that n satisfies $\varphi\,\mathcal{U}\,\psi$ i.e. suppose we are at time n and $\varphi\,\mathcal{U}\,\psi$ is true now.

   - Need it be the case that φ is true now? Either prove that n satisfies φ or give a counterexample.
   - Need it be the case that ψ is false now? Either prove that n satisfies $\neg\psi$ or give a counterexample.
   - Prove that n satisfies $\bigcirc\varphi \vee \bigcirc\psi$.
   - Show that φ does not need ever to become false, i.e. give a counterexample to demonstrate that it need not be the case that n satisfies $\langle F\rangle\neg\varphi$.

6. Recall the definitions of $\Box$ ("always") and $\Diamond$ ("sometimes"). Show that n satisfies $\Diamond\varphi$ iff n satisfies $\neg\Box\neg\varphi$, i.e. show that n satisfies $\langle P\rangle\varphi\vee\varphi\vee\langle F\rangle\varphi$ iff n satisfies $\neg([P]\neg\varphi\wedge\neg\varphi\wedge[F]\neg\varphi)$.