2D Transformations

COSC342

Lecture 4
8th March 2016

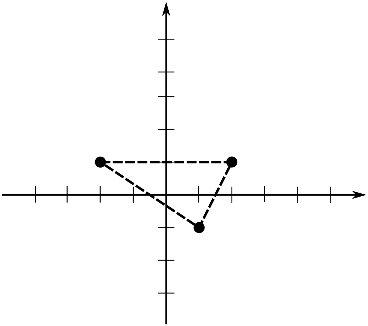
So What's This All About?

- Vectors and points in 2D
- Transformations - rotation, translation, scale, etc.
- Homogeneous co-ordinates
- Combining transformations
- Inverse transformations
- Directions and 'Points at Infinity'

Vectors and Points in 2D

- A useful representation of the point (x, y) is the vector $\begin{bmatrix} x \\ y \end{bmatrix}$
- Lines, polygons, etc. can then be represented as collections of vectors
- This can also be interpreted as the vector from the origin to the point

$$\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

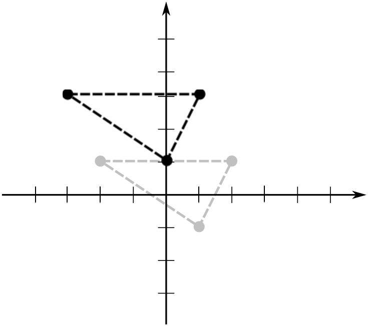


Transformations – Translation

- Shifting the point (x, y) by some offset $(\Delta x, \Delta y)$
- The point moves to $(x + \Delta x, y + \Delta y)$, which can be written as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

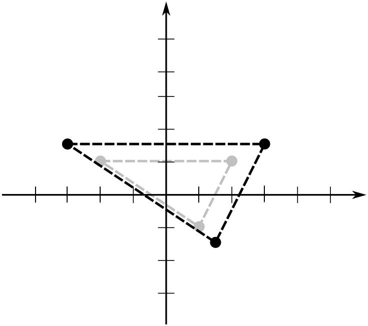


Transformations – Scaling

- Scaling points by some constant factor, s
- The point (x, y) moves to (sx, sy), which in vector terms is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = s \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = 1.5 \begin{bmatrix} x \\ y \end{bmatrix}$$

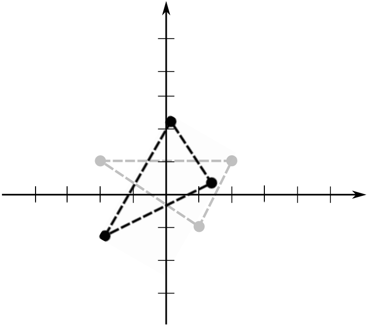


Transformations – Rotation

- Rotation by some angle, θ, around the origin
- This is expressed using a rotation matrix, R_θ:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathrm{R}_{60^\circ} \begin{bmatrix} x \\ y \end{bmatrix}$$



Inverse Transformations

- Transformations can be undone geometrically
  - The inverse of shifting by $(\Delta x, \Delta y)$ is shifting by $(-\Delta x, -\Delta y)$
  - The inverse of scaling by s is scaling by $1/s$
  - The inverse of rotation by θ is rotation by $-\theta$
- The inverse of a rotation matrix is its transpose:

$$\mathrm{R}_{-\theta} = \begin{bmatrix} \cos(-\theta & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \mathrm{R}_{\theta}^{T}$$

Combining Transformations

- Suppose we want to rotate 45° about the point $(1, 2)$:
  - We first shift by $(-1, -2)$ so that we're rotating about the origin
  - We then multiply by R_{45°
  - We then shift by $(1, 2)$ to undo the first translation
- This gives us

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \left(\mathrm{R}_{45^\circ} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right) \right) + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- It would be nice if there was an easy way to combine transforms

Homogeneous Co-ordinates

- The solution is somewhat counter-intuitive
- We represent 2D points as families of 3-vectors

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow k \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, k \neq 0$$

- The vector $\begin{bmatrix} a & b & c \end{bmatrix}^T$ corresponds to the point $(a/c, b/c)$
- These are known as homogeneous co-ordinates
- Now all the basic transformations become 3×3 matrices

Homogeneous Transformations – Translation and Rotation

- Translation by $(\Delta x, \Delta y)$ becomes

$$\begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + \Delta x \\ y + \Delta y \\ 1 \end{bmatrix}$$

- Rotation by an angle, θ, about the origin becomes

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta)x - \sin(\theta)y \\ \sin(\theta)x + \cos(\theta)y \\ 1 \end{bmatrix}$$

Homogeneous Transformations – Scaling

- Scaling by s becomes

$$\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} sx \\ sy \\ 1 \end{bmatrix}$$

- Sometimes this is represented as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/s \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1/s \end{bmatrix} \equiv \begin{bmatrix} sx \\ sy \\ 1 \end{bmatrix}$$

Combining Homogeneous Transformations

- Because all operations are now matrices we can combine them
- Suppose we have a series of transforms $\mathrm{T}_1, \mathrm{T}_2, \ldots, \mathrm{T}_k$
- Applying them to a point, $\mathbf{p}$, in order gives us

$$\left(\mathrm{T}_k \left(\mathrm{T}_{k-1} \ldots \left(\mathrm{T}_2 \left(\mathrm{T}_1 \mathbf{p}\right)\right)\right)\right)$$

- Since matrix multiplication is associative this is the same as

$$\left(\mathrm{T}_k \mathrm{T}_{k-1} \ldots \mathrm{T}_2 \mathrm{T}_1\right) \mathbf{p}$$

- We can combine the transforms once then apply it to a set of points

Combining Homogeneous Transformations

- Rotating 45° about the point $(1, 2)$:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{1/2} & -\sqrt{1/2} & 0 \\ \sqrt{1/2} & \sqrt{1/2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{1/2} & -\sqrt{1/2} & 1 + \sqrt{1/2} \\ \sqrt{1/2} & \sqrt{1/2} & 2 - 3\sqrt{1/2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Get a computer to do the arithmetic!

Inverse Homogeneous Transformations

- The inverse of translation by $(\Delta x, \Delta y)$ is translation by $(-\Delta x, -\Delta y)$

$$\begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -\Delta x \\ 0 & 1 & -\Delta y \\ 0 & 0 & 1 \end{bmatrix}$$

- The inverse of scaling by s is scaling by $1/s$

$$\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/s & 0 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix}$$

Inverse Homogeneous Transforms

- The inverse of rotation by θ is rotation by $-\theta$

$$\mathrm{R}_\theta^{-1} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{R}_\theta^T$$

- If we have a sequence of transforms, $\mathrm{T}_1, \mathrm{T}_2, \ldots \mathrm{T}_k$, then

$$(\mathrm{T}_k \mathrm{T}_{k-1} \ldots \mathrm{T}_2 \mathrm{T}_1)^{-1} = \mathrm{T}_1^{-1} \mathrm{T}_2^{-1} \ldots \mathrm{T}_{k-1}^{-1} \mathrm{T}_k^{-1}$$

- The individual transforms can be inverted geometrically

Locations and Directions

- A vector, $\mathbf{v} = [x, y]$, can be thought of as:
  - A *location* – the point at (x, y)
  - A *direction* – moving x units horizontally and y units vertically
- When we write $\mathbf{p}' = \mathbf{p} + \mathbf{d}$, we often mean moving the *point* $\mathbf{p}$ along the *direction* $\mathbf{u}$ to a new *location* $\mathbf{p}'$.
- In homogeneous co-ordinates we can distinguish between the two:

$$\mathbf{p} \equiv \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \mathbf{d} \equiv \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

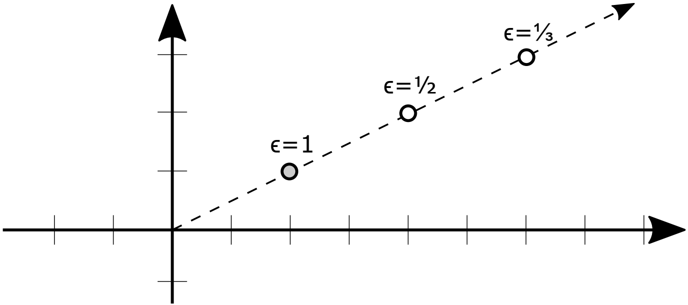
- But what does it mean to have the last element equal to 0?

'Points at Infinity'

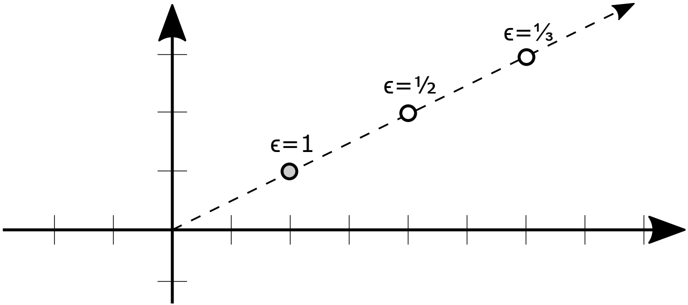
- If you think about what happens to the homogeneous point

$$\mathbf{v} \equiv [x, y, \epsilon]^{\mathsf{T}}$$

  as ϵ starts at 1 and heads towards 0...
- It starts at (x,y) then moves directly away from the origin...
- And we can make it as far away as we please by making ϵ small



Points at Infinity as Directions



- So you can think about $\mathbf{d} = [x, y, 0]^\mathsf{T}$ as a pure direction
- This is a bit like ‘north’ or ‘left’ or ‘up’
- You can go 10 m up, but ‘up’ is not 10 m away
- What happens if you transform a direction?