3D Transformations

COSC342

Lecture 5
13 March 2018

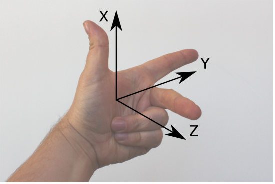
So What's This All About?

- Generalisation of the ideas from 2D
- Homogeneous co-ordinates etc.
- Transformations in 3D
- Rotations in 3D
  - Rotation about the X-, Y- or Z-axis
  - Rotations about an arbitrary axis

3D Co-ordinates

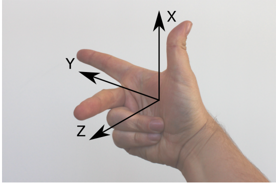
- We now need to move from 2D to 3D
- Many ideas are much the same as in 2D:
  - Homogeneous co-ordinates
  - Scaling and translation
- Some things are more complicated:
  - We have a choice of 'left-handed' or 'right-handed' co-ordinates
  - Rotations get much more complex
- One important new thing:
  - Projection from 3D to 2D (next lecture)

Left- and Right-Handed Co-ordinates

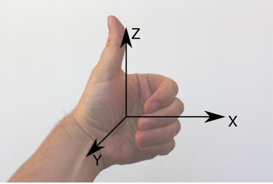


Left-handed

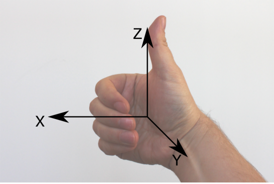
Thumb is X-axis
Forefinger is Y-axis
Middle finger is Z-axis



Right-handed



Fingers curl from X-axis to Y-axis
Thumb points along Z-axis



Homogeneous Co-ordinates in 3D

- In 3D we represent the point (x, y, z) as the family of 4-vectors

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to k \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, k \neq 0$$

- The vector $\begin{bmatrix} a & b & c & d \end{bmatrix}^T$ corresponds to the point $({}^a\!/_d, {}^b\!/_d, {}^c\!/_d)$
- Directions are of the form $\begin{bmatrix} x & y & z & 0 \end{bmatrix}^{\mathsf{T}}$
- The basic transformations are now 4×4 matrices

Translation and Scaling in 3D

- Translation and scaling are simple extensions from 2D
- Translation by $(\Delta x, \Delta y, \Delta z)$ becomes

$$\begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + \Delta x \\ y + \Delta y \\ z + \Delta z \\ 1 \end{bmatrix}$$

- Scaling by a factor s becomes

$$\begin{bmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} sx \\ sy \\ sz \\ 1 \end{bmatrix} \equiv \begin{bmatrix} x \\ y \\ z \\ 1/s \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotation in 3D

- Rotation in 3D is much more complex than in 2D
- Rotation in 3D has 3 'degrees of freedom'
- There are may ways to think about this
  - We'll start with rotations around the X-, Y- and Z-axes
  - Yaw, pitch, and roll
  - Rotation by some angle about an arbitrary axis

Rotation About X, Y and Z

- We know that a 2D rotation matrix looks like this:

$$\mathrm{R}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- This tells us how x and y change when we rotate from X towards Y
- In 3D rotating from X to Y is rotation about the Z-axis
- The Z value is unchanged, so we get

$$\mathrm{R}_Z\mathbf{v} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \\ z \\ 1 \end{bmatrix}$$

Rotation About X, Y and Z

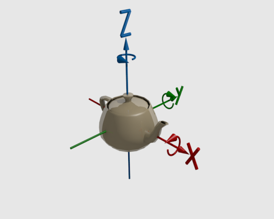
- We can think of rotation about the other axes in the same way
- Rotation about the X-axis is a rotation from Y to Z

$$\mathrm{R}_X\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y\cos(\theta) - z\sin(\theta) \\ y\sin(\theta) + z\cos(\theta) \\ 1 \end{bmatrix}$$

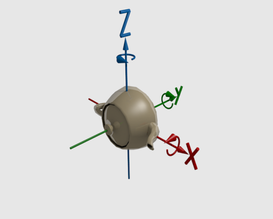
- Rotation about the Y-axis is a rotation from Z to X

$$\mathrm{R}_Y\mathbf{v} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos(\theta) + z\sin(\theta) \\ y \\ -x\sin(\theta) + z\cos(\theta) \\ 1 \end{bmatrix}$$

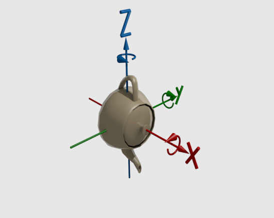
Rotation About X, Y and Z



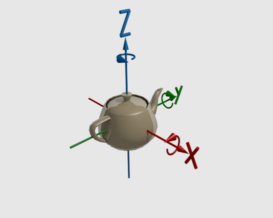
A teapot



Rotated by 90° around the X-axis



Rotated by 90° around the Y-axis



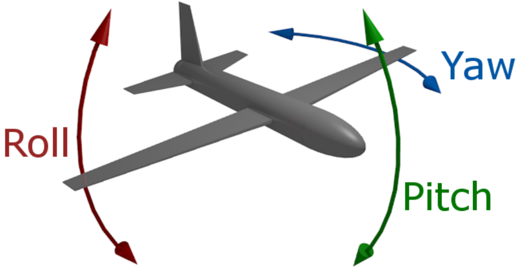
Rotated by 90° around the Z-axis

Gimbal Lock

- It turns out you can do any rotation with 3 angles
- These are called *Euler angles*
- There's a choice of axes and order
  - XYZ, YXZ, YZX, etc. (6 options), or
  - ZXZ, XYX, YXY, etc. (6 options)
- All options lead to *gimbal lock*
  - It is possible to rotate so that two axes are aligned
  - This removes one degree of freedom
  - You lose the ability to directly rotate in one axis

Roll, Pitch, and Yaw

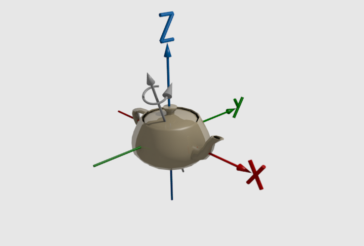
- Another way to think about rotation

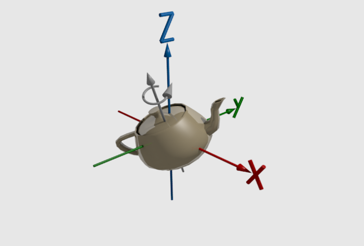


- Often these rotations are in terms of the object, not fixed axes

Rotation About an Axis

- Any 3D rotation can be expressed as rotation about a single axis





- Given an axis and an angle, how do we find a rotation matrix?
  1. Rotate the world so that the axis aligns with the X-axis
  2. Rotate by the required angle around the X-axis
  3. Undo the rotations from step (1)

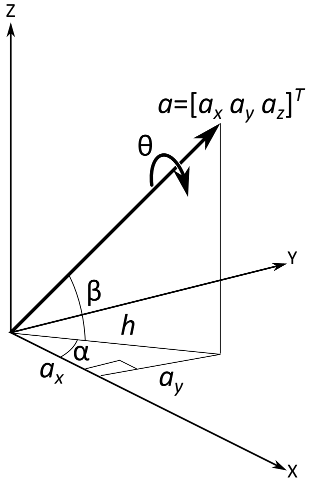
Rotation About an Axis

- The axis is $a = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}^T$
- First we rotate this so that it is in the X-Z plane
- We rotate about Z by $-\alpha$

$$h = \sqrt{a_x^2 + a_y^2}$$
$$\cos(\alpha) = \frac{a_x}{h}$$
$$\sin(\alpha) = \frac{a_y}{h}$$

- We don't need to solve for α

...

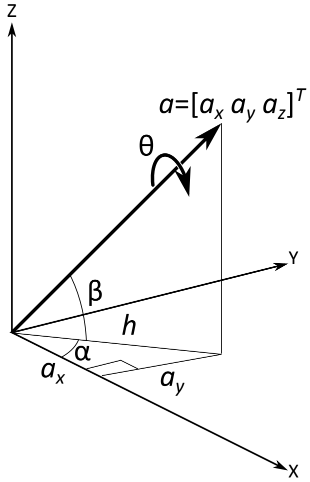


Rotation About an Axis

$$
\mathrm{A} = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

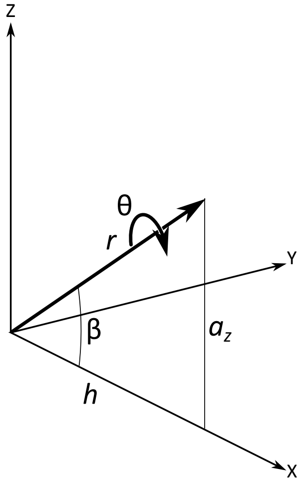
$$
= \begin{bmatrix} \frac{a_x}{h} & \frac{a_y}{h} & 0 & 0 \\ -\frac{a_x}{h} & \frac{a_x}{h} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



Rotation About an Axis

- Next we rotate about Y by $-\beta$

$$r = \sqrt{a_x^2 + a_y^2 + a_z^2}$$
$$\cos(\beta) = \frac{h}{r}$$
$$\sin(\beta) = \frac{a_z}{r}$$
$$\mathrm{B} = \begin{bmatrix} \frac{h}{r} & 0 & -\frac{a_z}{r} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{a_z}{r} & 0 & \frac{h}{r} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



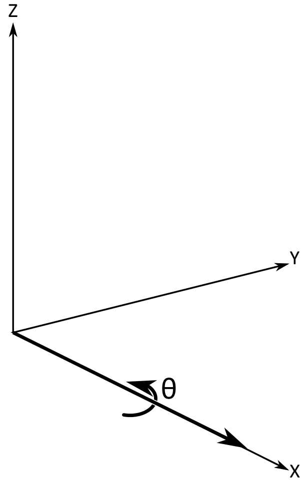
Rotation About an Axis

- Then rotate about X by θ

$$\mathrm{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Finally undo B then A
- The full transform is

$$\begin{aligned} \mathrm{T} &= \mathrm{A}^{-1}\mathrm{B}^{-1}\mathrm{CBA} \\ &= \mathrm{A}^{\mathsf{T}}\mathrm{B}^{\mathsf{T}}\mathrm{CBA} \end{aligned}$$



Quaternions

You may see *unit* quaternions used to represent rotations...

- $\mathbf{q} = q_r + q_x i + q_y j + q_z k$, where $q_r^2 + q_x^2 + q_y^2 + q_z^2 = 1$
- For rotation by angle θ around axis $\begin{bmatrix} a_x & a_y & a_z \end{bmatrix}^\mathsf{T}$:

$$q_r = \cos\left(\frac{\theta}{2}\right) \quad q_x = \sin\left(\frac{\theta}{2}\right) a_x \quad q_y = \sin\left(\frac{\theta}{2}\right) a_y \quad q_z = \sin\left(\frac{\theta}{2}\right) a_z$$

- i, j, k are imaginary values such that

$$i^2 = j^2 = k^2 = -1$$
$$ij = k \quad jk = i \quad ki = j \quad ji = -k \quad kj = -i \quad ik = -j$$

- Quaternions can be combined and applied to rotate points
- We won't be using them, so don't worry too much