# Wavelets: A Tutorial

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## Contents

# 1 Introduction

This is a tutorial on wavelets. Wavelets technology provides sophisticated mathematical and statistical routines for analysing signals and very large data sets. It originally had much to do with signal processing, but has recently been applied to many applications.

This tutorial will cover the basics of signal processing and how they are involved with wavelets. The Fourier analysis will be discussed and how it influenced the mathematical world. Wavelets in general will be discussed next and we will be focusing on the Haar Wavelet in detail with examples and implementations. Other types and forms of wavelets will be looked at, and lastly the tutorial will finish off with the applications of wavelets.

Because wavelets contain mathematical concepts, a mathematics overview section will be included in the appendix.

# 2 Signal Processing

This section introduces the reader to some basic signal processing knowledge.

Points to cover in this section:

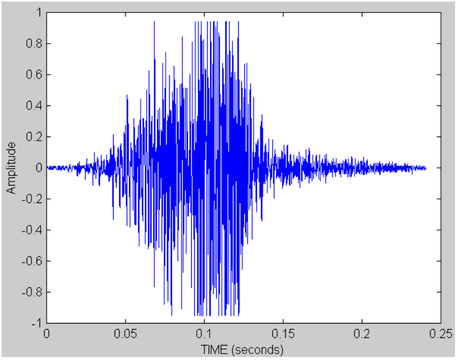


Figure 1: A signal of a dog's bark

- what are signals
- how do they relate to maths and wavelets
- time-domain signals
- frequency-domain signals

Signals are detectable physical quantity or impulse (as a voltage, current, or magnetic field strength) by which information may be sent. Common examples are sound signals or pulse signals. Signals can be represented digitally and shown as wave forms. An example is the signal of a dog's bark in Figure 1.

Mathematical transformations are applied to signals to highlight, reinforce, and to obtain further information from that signal that is not readily available in the raw signal. Frequency-domain signals and the Fourier transform are such transformations where its new representation are useful. They will be discussed in more detail later.

In signal processing, it is most common to find time-based signals. These are called *time-domain signals*, and are measured as a function of time. That is, when plotting the signal, one of the axes (normally the x-axis) is labelled 'time', and the other normally the amplitude of the signal. An example is shown in Figure 2. The time-domain signal represents the original shape of the signal and has not undergone any transformation. Although a popular and simple representation for a signal, it is not always the most useful representation for many growing applications.

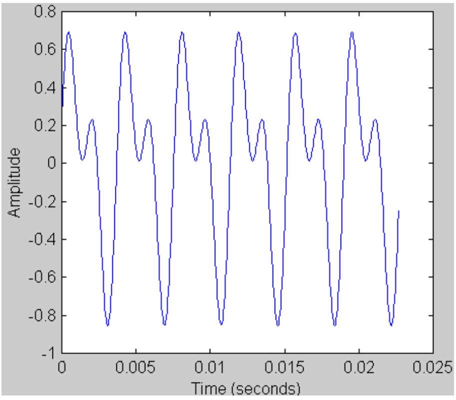


Figure 2: A time-domain signal.

*Frequency* is defined as the number of oscillation per second. The *frequency spectrum* shows what frequencies exists in the signal and instead of time, is measured in Hertz/Hz (cycles per second). An example is shown in Figure 3. The frequency content of the signals are used in many areas including engineering applications, and also in medical applications like the cardiologists who knows the healthy shape of the frequency of the biological signal of a patient's heart.

# 3 Fourier

This section briefly explains about the Fourier technology.

Points to cover in this section:

- Fourier synthesis
- Fourier Transform
- Disadvantages of Fourier

Mathematicians and scientists have long tried to create functions that can accurately model the curves and shape of signals. In the early 19th Century, Joseph Fourier came up with his well known theory about using sums of sin and cosine curves as an approximation for signals, known as the *Fourier synthesis*. Fourier's claim became an important role in the evolution of the ideas mathematicians had about functions.

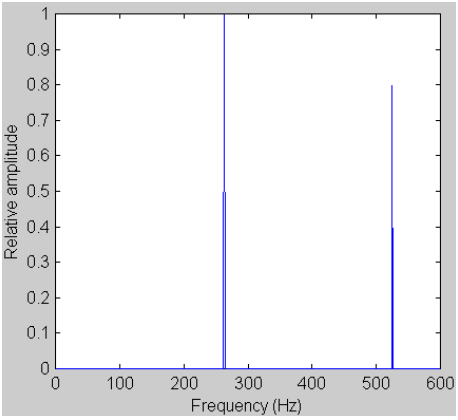


Figure 3: A frequency-domain signal of signal from Figure 2.

As mentioned before in section 2, mathematical functions are used to transform the raw signal into that of a useful one. The Fourier Transform's ability is in to translate a time-domain signal into a frequency-domain signal. Figure 3 is such an example of the Fourier transform. An *inverse Fourier transform* transforms data from the frequency domain back into the time domain. The detailed mathematics of the Fourier Transform is complicated and beyond the scope of this tutorial, however we will discuss the principle behind it:

> Any periodic signal is composed of a superposition of pure sine waves, with suitably chosen amplitudes and phases, whose frequencies are harmonics of the fundamental frequency of the signal.[13]

A *periodic signal* is a signal that exhibits a pattern that repeats with a regular time interval. A musical note is such an example. *Pure sine waves* are waves with only a single frequency associated with it. The word *superposition* just means that we add the amplitudes of two or more sine waves at a given time to get the total amplitude of the signal at that time. The *fundamental frequency* of the signal is calculated by measuring the time for the pattern to repeat (the period) and inverting that number to get a frequency in Hertz[1]. Finally, the word *harmonic* means that the frequency is an integer multiple of the fundamental frequency.

[1] The relationship between time and frequency in mathematical terms is given by the equation $f = 1/T$ where T is the period.

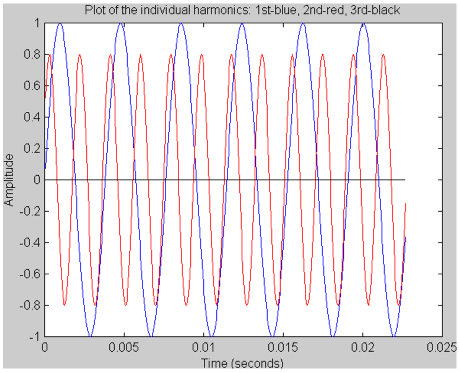


Figure 4: Graph of three inidividual pure harmonic signals.

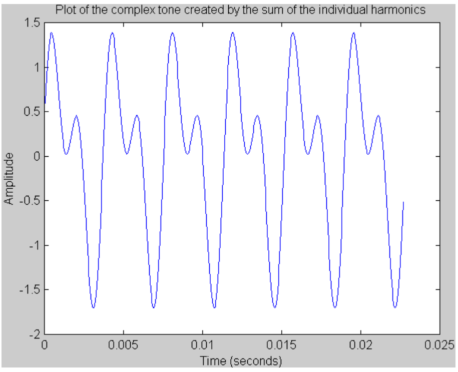


Figure 5: Result of the superposition of the individual signals in Figure 4

Figure 4 and Figure 5 illustrates this principle where the first figure shows the individual pure harmonic sine waves and the second figure shows the superposition of the waves to create a complex one.

However, the Fourier analysis works best with periodic signals, and with choppy signals with sharp peaks and drops, they can not be accurately modelled. Also, the Fourier transform/decomposition only showed the frequency representation of the signal, and thus losing any information regarding time. This became a big disadvantage, as it meant the Fourier transform could be used for only a few applications where the knowledge of time at the frequency was not needed.

# 4 Wavelets

This section discusses wavelets and its beginnings.

Points to cover in this section:

- wavelet definition and behaviour
- historical context

Since Fourier, scientists have wanted more appropriate functions than the sins and cosines which comprise the bases of Fourier analysis, to approximate choppy signals. Wavelets are a mathematical tool to decompose/transform data into different components and they offer the ability to represent the levels of details present in the data. Wavelets are well-suited for approximating data with sharp discontinuities. In wavelet analysis, the scale that we use to look at data plays a special role. Wavelet algorithms process data at different scales or resolutions.

In addition, wavelets transform are able to represent the signal simultaneously in both the frequency and time domain. An example for applications that uses this are Electroencephalograms (EEG), where the latency of an event-related potential is of particular interest [2]. Wavelet transform can be interpreted as a mapping of a signal in the time-domain $x(t)$ into a two-dimensional function of time and frequency. Essentially they decompose $x(t)$ into a family of functions, which are well localized in time (do not extend to infinity in time like the sin and cosine functions of a Fourier transform). This is done by choosing a "mother" wavelet $W(x)$. This is discussed in further detail later.

The wavelet technology is being used in many applications, mentioned later in this tutorial in the Applications section in section 7.

The first mention of wavelet appeared in A. Haar's thesis in 1909. By the 1930s, several groups working independently researched the representation of functions using scale-varying basis functions (explained later) which is the key to understanding wavelets. Later that year, Paul Levy [4] a physicist, used a scale-varying basis function called the Haar basis function to investigate Brownian motion (a type of random signal). He found the Haar basis function more advanced than those of the Fourier basis functions for his study of small

[2]Event-related potential is the response of the brain to a specific stimulus like flash-light, the latency of this response is the amount of time elapsed between the onset of the stimulus and the response.

complicated details. By the early 1980s, David Marr [5] developed an effective algorithm for numerical image processing using wavelets. Between 1960 and 1980, the mathematicians Guido Weiss and Ronald R. Coifman[5] studied the simplest elements of a function space, called atoms. In 1980, Grossman and Morlet[5], a physicist and an engineer, broadly defined wavelets in the context of quantum physics. In 1985, Stephane Mallat[5] gave wavelets an additional jump-start through his work in digital signal processing.

# 5 Haar Wavelet

This section examines the Haar wavelet[10][12] in detail, the oldest and simplest wavelet. The Haar transform uses square pulses to approximate the original function. We will discuss the basis functions and transforms for one-dimensional and two-dimensional Haar Wavelet. But first, basis functions and scale-varying basis functions will be explained.

## 5.1 What are Basis functions

In a digital perspective, every two-dimensional vector $(x, y)$ is a combination of the vector $(1, 0)$ and $(0, 1)$. These two vectors are the basis vectors for $(x, y)$. This is because that $x$ multiplied by $(1, 0)$ results in the vector $(x, 0)$ and $y$ multiplied by $(0, 1)$ is the vector $(0, y)$. The sum of the resulting vectors is $(x, y)$. For more examples on basis, see the appendix.

A property that is desirable is the basis vectors to be orthogonal or perpendicular to each other. For the basis $(1, 0)$ and $(0, 1)$, this criteria is satisfied. Another desirable property is *normalization*. This is discussed in detail in the appendix.

We go to the analog perspective to discuss how they relate to basis functions. Instead of the vector $(x, y)$, we have a function $f(x)$. If we imagine that $f(x)$ is a musical tone, we can construct A by superpositioning sin and cosine waves as discussed in Section 3. The sins and cosines are the basis functions in this example, and the elements of Fourier synthesis. For the sins and cosines chosen, we can set the additional requirement that they be orthogonal. This can be achieved by choosing the appropriate combination of sin and cosine function terms whose inner product add up to zero[3]. The

[3] The result of zero for inner product means the two functions are orthogonal to each other. See the appendix for more explanation.

particular set of functions that are orthogonal and that construct $f(x)$ are our orthogonal basis functions for this problem[4].

## 5.2 What are Scale-Varying Basis Functions

A basis function varies in scale by chopping up the same function or data space using different scale sizes. For example, we can imagine we have a signal over the domain from 0 to 1, denoted as $V^0$. We can divide the signal with two step functions that range from 0 to 1/2 and 1/2 to 1, denoted as $V^1$. Then we can divide the original signal again using four step functions from 0 to 1/4, 1/4 to 1/2, 1/2 to 3/4, and 3/4 to 1 denoted as $V^2$. And so on. Each set of representations code the original signal with a particular resolution or scale.

## 5.3 One-dimensional Haar Wavelet

### 5.3.1 Basis Functions

The Haar wavelet's scale varying functions are usually denoted by the symbol $\phi$. A simple basis for these scaling functions $V^j$ are given by the set of scaled and translated "box" functions':

$$\phi_i^j(x) := \phi(2^j x - i), \qquad i = 0, ..., 2^j - 1,$$

where,

$$\phi(x) := \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

The graph of $V^2$ is shown in Figure 6.

We can now define a new vector space $W^j$ as the orthogonal complement of $V^j$ on $V^{j+1}$. In another words, we will let $W^j$ be the space of all functions in $V^{j+1}$ that are orthogonal to all functions in $V^j$ under the chosen inner product.

A collection of linearly independent functions $\psi_i^j(x)$ spanning $W^j$ are called *wavelets*.

The wavelets corresponding to the box basis are known as the Haar wavelets, given by:

$$\psi_i^j(x) := \psi(2^j x - i), \qquad i = 0, ..., 2^j - 1,$$

[4]Note not all wavelets have this orthogonal property.

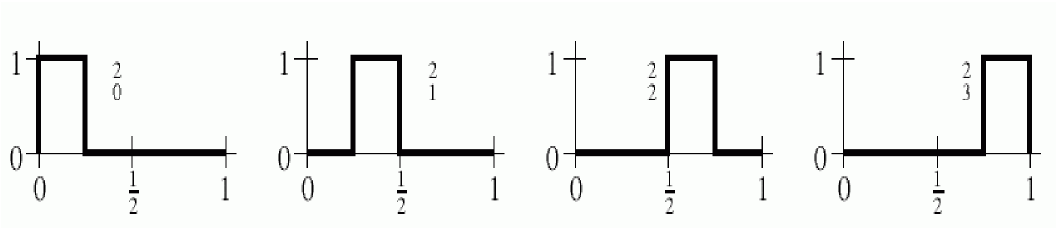

Figure 6: Scaling basis functions for $V^2$.

where

$$\psi(x) := \begin{cases} 1 & \text{for } 0 \leq x < 1/2 \\ -1 & \text{for } 1/2 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

The graph for Haar wavelets for $W^1$ is shown in Figure 7.

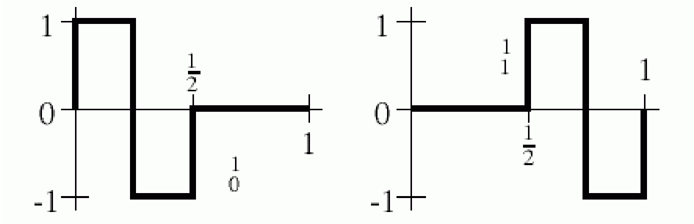

Figure 7: Haar Wavelets for $W^1$.

### 5.3.2 Transform

To compute the Haar wavelet transform (also called the *wavelet decomposition*), the data is recursively averaged and differenced. The result is a single coefficient representing the overall data followed by other coefficients (in increasing resolution order), which correspond to how to reconstruct back to the original data. These coefficients are called the *detail coefficients*. Thus, the detail coefficients are the coefficients of the wavelet basis functions.

A use of a one-dimensional Haar wavelet transform can be shown by an example of a one-dimensional image with just 4 pixels:

$$[9\ 7\ 3\ 5]$$

By averaging the numbers in pairs, we get:

[8 4]

In this representation, there is no lose or gain in information of the data. But the lose in information makes a lose in the detail. To restore to the original detail, its detail coefficient is also stored. In this example, its detail coefficient is:

[1 -1]

as 9 - 8 = 1 which corresponds to the first pair of original pixel values. For the second pair, 3 - 4 is -1. So at the current detail, the coefficients are:

[8 4 1 -1]

If we continue with the recursive averaging and differencing, the

[8 4] now becomes [6]

and its detail coefficients are

[2] (8 - 6 = 2)

So the wavelet transform with the single overall average coefficient followed by detail coefficients in increasing order is

[6 2 1 -1]

The power of this wavelet transform is that one can construct the pixels to any resolution/detail by recursively adding and subtracting the detail coefficients from lower resolution pixels. Also in storing the wavelet transform than that of the original data, the detail coefficients tend to be much smaller in magnitude.

So given the transform, one can reconstruct the data to any resolution by recursively adding and subtracting the detail coefficients from lower resolution versions. To shorten or to removing these coefficients from this representation results in only small errors in the reconstructed image, and can be used in image compression.

For example, from the [6 2 1 -1] to get to the higher resolution [8 4 1 -1], 6 is added to 2, and then subtracted by 2 to get the [8] and [4] part respectively. And to get to the original [9 7 3 5] from [8 4 1 -1], 8 is added to 1 and subtracted by 1 to get 9 and 7 respectively. and for the 3 and 5, they are attained by 4 + (-1), and 4 - (-1) respectively.

A program implementation of the one-dimensional Haar transform is included in the appendix.

### 5.4 Two-dimensional Haar Wavelet

#### 5.4.1 Basis

The standard construction of a two-dimensional wavelet basis is complicated and beyond the scope of this tutorial. The interested reader may want to read up G. Potts and W. Shoaff's "Texture Mapping with Wavelet Transforms" [6] for a explanation of how to build these basis.

#### 5.4.2 Transform

To obtain the standard decomposition of an 2-dimensional image, each *row* of pixel values are transformed using the one-dimensional wavelet transform. This gives a average value along with detail coefficients for each row in the image. The next step is apply the one-dimensional transform to each *column*, giving a single average value of the image in the top left corner coefficient, and the rest of the data being detail coefficients.

#### 5.4.3 Example of Image compression using the Haar wavelet transform.

An image can be represented as a two-dimensional array. The procedure to achieve the wavelet transform is using the averaging and differencing until a new matrix representation is achieved. The new matrix is a more concise form. To achieve the compression, some small insignificant data is removed, and the averaging and differencing process reversed to arrive at the approximated original image at different detail level. The examples are shown from Figure 8 to Figure 11. More examples can be found from Peggy Morton's webpage[9] and Colm Mulcahy's easy-to-read but detailed paper[11]. A very mathematically detailed explanation of the Haar transform and basis can be found in Kristian Sandberg's site[8].

## 6 The Daubechies Wavelets

Haar wavelets are old. They were invented in the early 20th century. Their graphs are made from flat pieces. Although an improvement to Fourier synthesis, approximation to most signals are poor, and to represent a sloping line, many flat pieces are needed for good accuracy. The basis will not give compression ratios of 20:1 or 100:1 as sometimes desired.

$$M=\begin{bmatrix} 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \\ 9 & 55 & 54 & 12 & 13 & 51 & 50 & 16 \\ 17 & 47 & 46 & 20 & 21 & 43 & 42 & 24 \\ 40 & 26 & 27 & 37 & 36 & 30 & 31 & 33 \\ 32 & 34 & 35 & 29 & 28 & 38 & 39 & 25 \\ 41 & 23 & 22 & 44 & 45 & 19 & 18 & 48 \\ 49 & 15 & 14 & 52 & 53 & 11 & 10 & 56 \\ 8 & 58 & 59 & 5 & 4 & 62 & 63 & 1 \end{bmatrix}$$

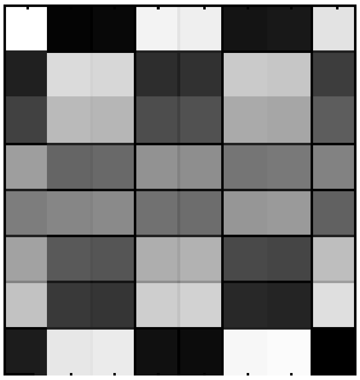

Figure 8: Image representation of the 8x8 array.

$$\begin{bmatrix} 32.5 & 0 & .5 & .5 & 31 & -29 & 27 & -25 \\ 32.5 & 0 & -.5 & -.5 & -23 & 21 & -19 & 17 \\ 32.5 & 0 & -.5 & -.5 & -15 & 13 & -11 & 9 \\ 32.5 & 0 & .5 & .5 & 7 & -5 & 3 & -1 \\ 32.5 & 0 & .5 & .5 & -1 & 3 & -5 & 7 \\ 32.5 & 0 & -.5 & -.5 & 9 & -11 & 13 & -15 \\ 32.5 & 0 & -.5 & -.5 & 17 & -19 & 21 & -23 \\ 32.5 & 0 & .5 & .5 & -25 & 27 & -29 & 31 \end{bmatrix}$$

Figure 9: First each rows are recursively averaged and differenced.

$$\begin{bmatrix} 32.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -4 & 4 & -4 \\ 0 & 0 & 0 & 0 & 4 & -4 & 4 & -4 \\ 0 & 0 & .5 & .5 & 27 & -25 & 23 & -21 \\ 0 & 0 & -.5 & -.5 & -11 & 9 & -7 & 5 \\ 0 & 0 & .5 & .5 & -5 & 7 & -9 & 11 \\ 0 & 0 & -.5 & -.5 & 21 & -23 & 25 & -27 \end{bmatrix}$$

Figure 10: Then each column recursively averaged and differenced.

In 1988, at AT&T Bell Laboratories, Ingrid Daubechies[14] developed the now improved Daubechies wavelet. Figure 12 shows the Daubechies wavelet.

### 6.1 Other Wavelets

There are many other wavelets each with different characteristics. These include:

- Coiflets
- Symlets
- BiorSplines
- Meyer
- Morlet
- Mexican_hat
- Complex Gaussian
- Shannon

## 7 Applications

This section shows how the wavelets technology have been utilized in different fields and applications.

### 7.1 JPEG2000

The latest version of JPEG compression contains algorithms using wavelets.

With current JPEG algorithms forming unpleasant block particularly at colour transitions, JPEG experts are developing JPEG2000, a new image coding system using wavelets. Instead of using blocks of pixels, they are replaced with mathematical functions, which represent the image information in the form of oscillation curves. The differences in colour and brightness are more accurately taken into account, and colour changes are smoother.

The positive side-effects of this method:

- Individual areas of the image can be excluded from compression.
- An image can be built up on the screen in steps, from coarse to increasingly fine resolution.
- Additional information, such as a digital watermark, is easier to integrate.

Though an improved algorithm, its takes longer computation time to implement.

The new compression method is already used in the LuraWave Plugin for Adobe Photoshop, Paint Shop Pro and ACDSee.

While testing, at a degree of compression corresponding to 100 percent quality preservation in JPEG, one can see far fewer "staircases" along the color contrast lines in JPEG2000 format than when using its predecessor.

More information concerning the JPEG2000 technology can be found at their website[1].

## 7.2 Computer and Human Vision

In the early 1980s, David Marr[15] began work at MIT's Artificial Intelligence Laboratory on artificial vision for robots. He was an expert on the human visual system and his goal was to learn why the first attempts to construct a robot capable of understanding its surroundings were unsuccessful[16].

Marr believed that it was important to establish scientific foundations for vision, and that while doing so, one must limit the scope of investigation by excluding everything that depends on training, culture, etc, and focus on the mechanical or involuntary aspects of vision. This low-level vision is the part that enables us to recreate the three-dimensional organization of the physical world around us from the excitations that stimulate the retina. Marr asked the questions:

- How is it possible to define the contours of objects from the variations of their light intensity?
- How is it possible to sense depth?
- How is movement sensed?

Marr then went to develop working algorithmic solutions to the above questions.

Marr's theory was that image processing in the human visual system has a complicated hierarchical structure that involves several layers of processing. At each processing level, the retinal system provides a visual representation that scales progressively in a geometrical manner. His arguments hinged on the detection of intensity changes. He theorized that intensity changes occur at different scales in an image, so that their optimal detection requires the use of operators of different sizes. He also theorized that sudden intensity changes produce a peak or trough in the first derivative of the image. These two hypotheses require that a vision filter have two characteristics: it should be a differential operator, and it should be capable of being tuned to act at any desired scale. Marr's operator was a wavelet that today is referred to as a "Marr wavelet."

## 7.3 FBI Fingerprint Compression

The US Federal Bureau of Investigation uses wavelets to compress its growing database of over two hundred million fingerprint records.

Old methods were done with inked impressions on paper cards, while facsimile scans of the impressions were distributed among law enforcement agencies, but digitization quality is low. Even with experimenting of digital storage of the prints, incompatibilities between data formats have recently become a problem. This problem led to a demand in the criminal justice community for a digitization and a compression standard.

Reconstruction on right image in Figure 13 uses only the largest 5 percent of the coefficients. The original image is the left image.

## 7.4 Others

Wavelets are used in other applications including:

- Finance and economics - multi-resolution time-series analysis
- Statistics - denoising, robust estimation, non-parametric regression and smoothing
- Geophysics - seismic data processing

- Mechanical and aerospace engineering - fluid dynamics and turbulence modelling
- Speech and audio processing - speech compression, voice recognition, speaker verification and sound synthesis

# 8 Conclusion

The purpose of this tutorial was to give readers a brief overview of the wavelets technology, in particular the Haar wavelet. A brief overview of signal processing followed by the Fourier synthesis were discussed and how it affected the mathematical world. Basis functions and scale varying basis functions were explained, and the transform and basis functions were explained for both the one-dimensional and two dimensional Haar wavelet. Further applications of the wavelet technology were discussed, including FBI image compression and computer vision.

There are many more Wavelet resources for the interested reader including papers and websites on wavelets[5][10] [12] and source listing[7], papers on the history of Fourier[3] and his famous claims[2].

# 9 Appendix

## 9.1 Mathematical Background

This section provides the reader with the mathematical knowledge that will help understanding the foundation of wavelets. Most of the definitions are found from H. Anton and C. Rorres book "Elementary Linear Algebra"[17]. More examples can be found there.

### 9.1.1 Linear Independence

Definition 1: If $S = \{v_1, v_2, \cdots, v_r\}$ is a nonempty set of vectors, then the vector equation:

$$k_1 v_1 + k_2 v_2 + \cdots + k_r v_r = 0 \tag{1}$$

has at least one solution, namely

$$k_1 = 0, k_2 = 0, \cdots, k_r = 0 \tag{2}$$

If this is the only solution, then $S$ is called a linearly independent set.
DEFINITION 2: A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.
**Example 1**: If $v_1 = (2, -1, 0, 3), v_2 = (1, 2, 5, -1)$, and $v_3 = (7, -1, 5, 8)$, then the set of vectors $S = \{v_1, v_2, v_3\}$ is linearly dependent, since $3v_1 + v_2 - v_3 = 0$.
**Example 2**: The polynomials:

$$p_1 = 1 - x, p_2 = 5 + 3x - 2x^2, andp_3 = 1 + 3x - x^2$$

form a linearly dependent set in $P_2$ since $3p_1 - p_2 - 2p_3 = 0$.

### 9.1.2 Span

DEFINITION: If $S = \{v_1, v_2, \cdots, v_r\}$ is a set of vectors in a vector space $V$, then the subspace $W$ of $V$ consisting of all linear combinations of the vectors in $S$ is called the space spanned by $v_1, v_2, \cdots, v_r$, and we say that the vectors $v_1, v_2, \cdots, v_r$ span $W$.

### 9.1.3 Basis for a Vector Space

DEFINITION: If $V$ is any vector space and $S = \{v_1, v_2, \cdots, v_n\}$ is a set of vectors in $V$, then $S$ is called a basis for $V$ if the following two conditions hold:

- $S$ is linearly independent
- $S$ spans $V$

In words, each vector in the space can be expressed in one way only as a expressed as combination of the basis vectors.
**Example**: For the 2-dimension, every vector $(x, y)$, is the combination of the basis vectors $(1, 0)$ and $(0, 1)$. The vector $(4, 2)$, is 4 times the first basis plus 2 times the second basis. Another pair of basis is $(1, 1)$ and $(1, -1)$. Again they produce all plane vectors including the vector $(4, 2)$. However $(1, 0)$ and $(2, 0)$ would not be a basis; no combination will result in $(4, 2)$.

### 9.1.4 Inner Product

**Defintion**: An inner product on a real vector space $V$ is a function that associates a real number $\langle u, v\rangle$ with each pair of vectors $u$ and $v$ in $V$ in such a way that the following axioms are satisfied for all vectors $u$, $v$ and $w$ in $V$ and all scalars $k$.

1. $\langle u, v\rangle = \langle v, u\rangle$ [Symmetry axiom]
2. $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ [Additivity axiom]
3. $\langle ku, v\rangle = k\langle u, v\rangle$ [Homogeneity axiom]
4. $\langle v, v\rangle \geq 0$ [Positivity axiom]
   where $\langle v, v\rangle = 0$
   if and only if $v = 0$.

The standard inner product equation is:

$$\langle f|g\rangle := \int_0^1 f(x)g(x)dx$$

## 9.2 Program Implementation and Comments

The following code is the normalized Haar wavelet transform in one-dimensional. It is written in C.

### 9.2.1 Program Code

```
#include <math.h>

#define MAXARRAYSIZE 20

float val[8];
int height_pic, width_pic;

int noElements;
float array[MAXARRAYSIZE];

// prints values in array
print_c() {
```

```
  int i;

  printf("Value of c: ");
  for (i = 1; i <= noElements; i++)
    printf("%f ", array[i]);
  printf("\n");
}

int decompositionStep(float array[], int h) {
  int i, j;
  float s;
  float val2[MAXARRAYSIZE];

  // printf("h value is: %d\n", h);

  s = sqrt(2);
  for (i = 1; i <= h/2; i++) {
    val2[i] = (array[2*i - 1] + array[2*i])/s;
    printf("val[%d] at i=%d is %f\n", i, i, val2[i]);

    val2[h/2 + i] = (array[2*i -1] - array[2*i])/s;
    printf("val[%d] at i=%d is %f\n", (h/2 + i), i, val2[i]);
  }

  //val = val2;
  for (i = 1; i <= MAXARRAYSIZE; i++)
      array[i] = val2[i];
  printf("After for loop, ");
  print_c();

  return 0;
}

int decomposition() {
  float s;
  int i, j;

  array[1] = 9;
```

```
  array[2] = 7;
  array[3] = 3;
  array[4] = 5;

  noElements = 4;
  j = noElements;
  printf("Start\n");
  print_c();

  // val = val/sqrt(j); // normalization step
  s = sqrt(noElements);
  for (i = 1; i <= noElements; i++)
    array[i] = array[i]/s;

  while (j > 1) {
    decompositionStep(array, j);
    j = j/2;
  }



  return 0;
}

main() {
  decomposition();
}
```

### 9.2.2 Comments

*Normalization* is another property that is desirable in wavelets. A basis function $u(x)$ is normalized if $\langle u|u\rangle = 1$. We can normalize the Haar basis by replacing our earlier definitions with:

$$\phi_i^j := 2^{j/2}\phi(2^j x - i)$$

and

$$\psi_i^j(x) := 2^{j/2}\psi(2^j x - i)$$

where the constant factor of $2^{j/2}$ is chosen to satisfy $\langle u|u\rangle = 1$ for the standard inner product. The normalization step is shown in the code.
The implementation of Haar transform uses 4 coefficients. It can be extended to calculate a higher number of coefficients by changing the number of `noElements` and adding elements in the array, or by reading in a file.

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$$N=\begin{bmatrix} 32.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -4 & 4 & -4 \\ 0 & 0 & 0 & 0 & 4 & -4 & 4 & -4 \\ 0 & 0 & .5 & .5 & 27 & -25 & 23 & -21 \\ 0 & 0 & -.5 & -.5 & -11 & 9 & -7 & 5 \\ 0 & 0 & .5 & .5 & -5 & 7 & -9 & 11 \\ 0 & 0 & -.5 & -.5 & 21 & -23 & 25 & -27 \end{bmatrix}$$

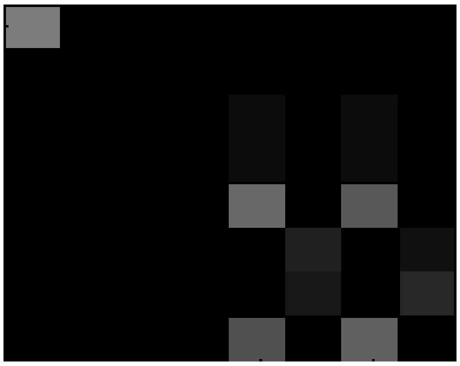

Figure 11: After averaging and differencing process. The top left value is the averaged value of the entire image, while the rest are the detail coefficients.

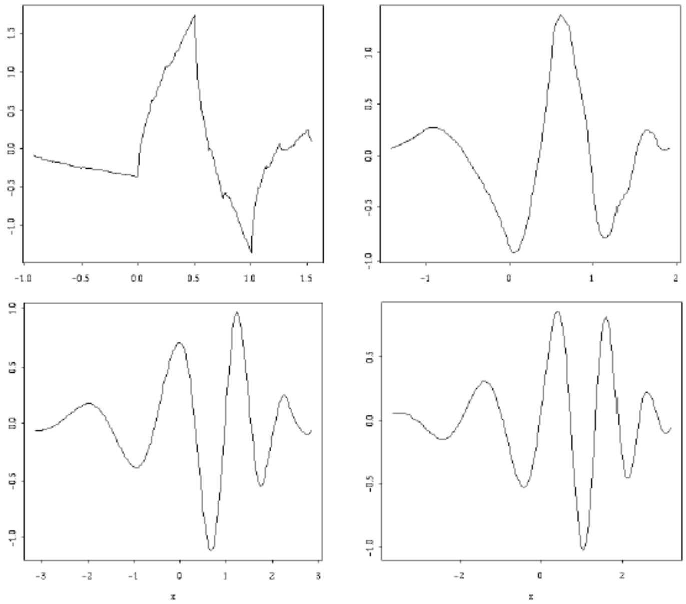

Figure 12: Daubechies Wavelets.

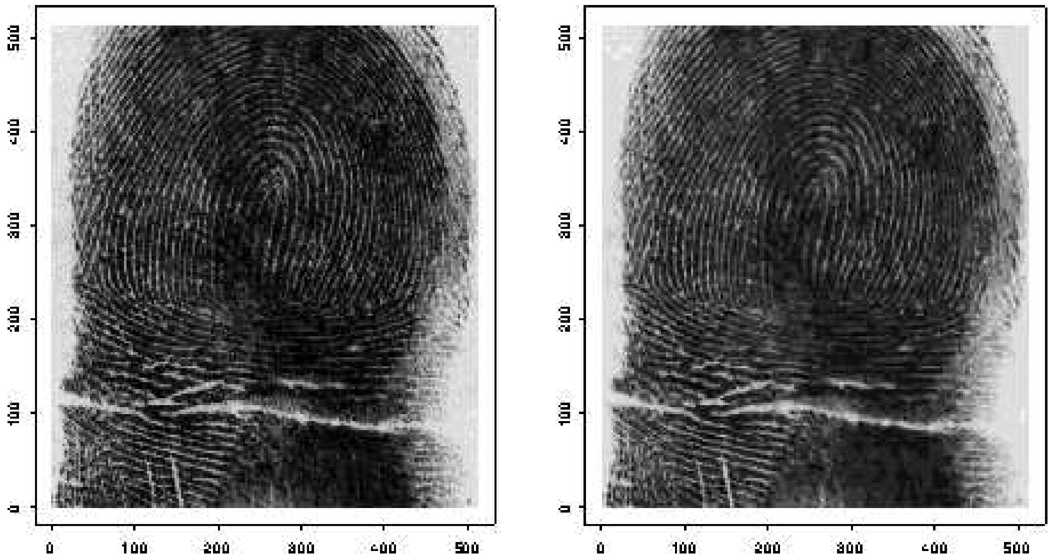

Figure 13: Wavelets used in fingerprints.