Concerning the Quadratic Relations which define the Grassman Manifold

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Abstract

The Plücker embedding gives a bijective correspondence between the d-planes of a projective space $\mathbb{P}^n$ and the points of the Grassman Manifold in the higher dimensional space $\mathbb{P}^N$, where $N = \binom{n+1}{d+1} -1$. The Grassman Manifold can be defined as the set of points in $\mathbb{P}^N$ whose homogeneous coordinates satisfy certain quadratic relations, those relations being generated by sequences in $\{0,...,n\}$. Here we present a minimal set of generating sequences for the quadratic relations and subsequently investigate the linear independence of said relations.

1 Introduction

When considering d-planes in a projective space $\mathbb{P}^n$, it is often useful to consider the Grassman Manifold in the higher dimensional space $\mathbb{P}^N$, where $N = \binom{n+1}{d+1} -1$. It is therefore useful to consider the quadratic relations which define the Grassman Manifold.

We begin with a brief discussion of projective space, d-planes in projective space, and the Plücker coordinates of a d-plane in projective space. We then see that the Plücker coordinates define a point in $\mathbb{P}^N$, that each such point satisfies the aforementioned quadratic relations, and that all such points comprise the Grassman Manifold.

To define a projective space $\mathbb{P}^n$, we consider the set $\mathbb{C}_*^{n+1} = \mathbb{C}^{n+1} - \{\mathbf{0}\}$ of nonzero points in complex $(n+1)$ space. We define an equivalence relation $\backsim$ on $\mathbb{C}_*^{n+1}$ by writing $(z_0,...,z_n) \backsim (w_0,...,w_n)$ if $(z_0,...,z_n) = c(w_0,...,w_n)$ for some $c \neq 0$. Then $\mathbb{P}^n$ is the set $\mathbb{C}_*^{n+1}/\backsim$. A point z in $\mathbb{P}^n$ is denoted $[z_0 : ... : z_n]$. The numbers $z_0,...,z_n$ are called the homogeneous coordinates of z.

A d-plane in $\mathbb{P}^n$ consists of all one-dimensional subspaces in the span of $(d+1)$ linearly independent vectors in $\mathbb{C}_*^{n+1}$ (a.k.a. points in $\mathbb{P}^n$). Alternatively, it is the set of points $z = [z_0 : ... : z_n]$ in $\mathbb{P}^n$ whose coordinates satisfy $(n-d)$ independent equations $\sum_{j=0}^{n} a_{\beta j} z_j = 0$, where $1 \leq \beta \leq (n-d)$.

Given a d-plane L in $\mathbb{P}^n$, we can obtain a vector in $\mathbb{C}_*^{N+1}$ by means of the Plücker embedding. That is, given $(d+1)$ linearly independent vectors

$\mathbf{z_0},...,\mathbf{z_d}$ in $\mathbb{C}_*^{n+1}$ which span L, we first form the $(d+1) \times (n+1)$ matrix $Z = \begin{pmatrix} (z_0)_0 & (z_0)_1 & \cdots & (z_0)_n \\ \vdots & \vdots & \ddots & \vdots \\ (z_d)_0 & (z_d)_1 & \cdots & (z_d)_n \end{pmatrix}$. We then consider the $(d+1) \times (d+1)$ submatrices of Z, which we can characterize completely by the set $J = \{j_0...j_d : 0 \leq j_0 < ... < j_d \leq n\}$ of strictly increasing sequences in $\{0,...,n\}$. (Each sequence $j_0,...,j_d$ in J represents the $(d+1) \times (d+1)$ submatrix of Z consisting of the $j_0th,...,j_dth$ columns of Z.) We order the elements of J lexicographically and denote the determinant of the $j_0...j_d$-th submatrix by $p(j_0...j_d)$. There are $\binom{n+1}{d+1}$ sequences in J, and, since $\mathbf{z_0},...,\mathbf{z_d}$ are linearly independent, we have $p(k_0...k_d) \neq 0$ for some $k_0...k_d$ in J. Thus, $(...,p(j_0...j_d),...)$ forms a vector in $\mathbb{C}_*^{N+1}$. The homogeneous coordinates of $[... : p(j_0...j_d) : ...]$ are known as the Plücker coordinates of the d-plane L.

To see that the Plücker embedding of L into $\mathbb{C}_*^{N+1}$ in fact defines the point $[... : p(j_0...j_d) : ...]$ in $\mathbb{P}^N$, we take $\mathbf{w_0},...,\mathbf{w_d}$ in $\mathbb{P}^n$ which span L. We construct W as we constructed Z previously, and we denote the determinant of the $j_0...j_d$-th submatrix of W by $q(j_0...j_d)$. By a change of basis, we see that $Z = CW$ for some $(d+1) \times (d+1)$ matrix C with $det(C) \neq 0$. Thus $p(j_0...j_d) = det(C)q(j_0...j_d)$ so that $[... : p(j_0...j_d) : ...]$ is indeed a point of $\mathbb{P}^N$.

We next arrive at Theorem 1 in [Kl&La72]. *The Plücker embedding gives a bijective correspondence between the d-planes in $\mathbb{P}^n$ and the points of $\mathbb{P}^N$ whose homogeneous coordinates satisfy the quadratic relations*

$$\sum_{\lambda=0}^{d+1} (-1)^\lambda p(j_0...j_{d-1}k_\lambda)p(k_0...\widehat{k_\lambda}...k_{d+1}) = 0,$$

where $j_0...j_{d-1}$ and $k_0...k_{d+1}$ are sequences with $0 \leq j_\beta, k_\gamma \leq n$ and $k_0...\widehat{k_\lambda}...k_{d+1}$ denotes $k_0...k_{d+1}$ with k_λ removed.

That is, the Plücker coordinates of each d-plane in $\mathbb{P}^n$ satisfy the quadratic relations, and conversely, a point $[z_0 : ... : z_N]$ of $\mathbb{P}^N$ whose homogeneous coordinates satisfy — as described below — the quadratic relations has the Plücker coordinates of a unique d-plane in $\mathbb{P}^n$. This is how the Grassman Manifold arises and why it is the set of points in $\mathbb{P}^N$ whose coordinates satisfy the quadratic relations. The Grassman Manifold (of d-planes in projective n-space) is denoted $G_{d,n}$.

To see how the homogeneous coordinates of a point $[z_0 : ... : z_N]$ of $\mathbb{P}^N$ can satisfy the quadratic relations, we first note that because p is a determinant, we have

$$\begin{aligned} p(j_0...j_d) &= 0 \text{ if any two of the } j_\beta \text{ are equal;} \\ p(j_0...j_d) &= -p(j_0...j_{\beta-1}j_{\beta+1}j_\beta...j_d) \text{ for } \beta = 0,...,d-1. \end{aligned}$$

Thus we can write any quadratic relation so that each sequence $l_0...l_d$ in each factor $p(l_0...l_d)$ of each term $p(l_0...l_d)p(m_0...m_d)$ is strictly increasing. If we now represent points $[z_0 : ... : z_N]$ of $\mathbb{P}^N$ using the ordered set J above in place of the set of indices $\{0 < 1 < ... < N\}$ and we identify $z_{j_0...j_d}$ with $p(j_0...j_d)$ we see that the quadratic relations express possible dependencies among the homogeneous coordinates $z_0,...,z_N$ of the points $[z_0 : ... : z_N]$ of $\mathbb{P}^N$. I.e. the homogeneous coordinates of a point $[z_0 : ... : z_N]$ can satisfy the quadratic relations.

In what follows, we investigate the quadratic relations. We consider the sequence pairs $j_0...j_{d-1}$ and $k_0...k_{d+1}$ which generate nontrivial quadratic relations. Evidently, there are many such pairs. Questions we answer are: How many different quadratic relations do said pairs generate? Is there a minimal set of these pairs that generates the quadratic relations? And are the quadratic relations themselves independent? I.e. can we write certain quadratic relations as linear combinations of other quadratic relations? Or is this impossible?

2 Notation and Preliminary Results

We begin with some notation. Any sequence $j_0,...,j_m$ is written $j_0...j_m$ or abbreviated $\mathbf{j}$. The set $\{j_0,...,j_m\}$ is denoted J and the number of elements in J is denoted $\#J$.

Given sequences $\mathbf{j}, \mathbf{k}$ we define $\mathbf{j} \cap \mathbf{k}, \mathbf{k} - \mathbf{j}$, and $\mathbf{j} \cup \mathbf{k}$ as follows:

•$\mathbf{j} \cap \mathbf{k}$ is the subsequence of $\mathbf{j}$ all of whose terms also belong to $\mathbf{k}$. Note that $\mathbf{j} \cap \mathbf{k}$ might be the empty sequence. I.e. $J \cap K$ might be $\emptyset$.

•$\mathbf{k} - \mathbf{j}$ is the subsequence of $\mathbf{k}$ all of whose terms do not belong to $\mathbf{j}$.

•$\mathbf{j} \cup \mathbf{k}$ is the sequence given by $\mathbf{j}(\mathbf{k} - \mathbf{j})$.

We denote a quadratic relation

$$\sum_{\lambda=0}^{d+1} (-1)^\lambda p(j_0...j_{d-1}k_\lambda)p(k_0...\widehat{k_\lambda}...k_{d+1}) = 0$$

by its left hand side. That is, $Q = \sum_{\lambda=0}^{d+1} (-1)^\lambda p(j_0...j_{d-1}k_\lambda)p(k_0...\widehat{k_\lambda}...k_{d+1})$ is said to be a quadratic relation. The sequences $j_0,...,j_{d-1}$ and $k_0,...,k_{d+1}$ are said to generate Q.

In addition, a quadratic relation Q is assumed to be written in simplest form. That is, terms in Q which are alike with respect to the sequences $j_0...j_{d-1}, k_0...k_{d+1}$ are assumed to have been combined and/or cancelled. If for example $Q = -p(01)p(02) + p(02)p(01)$ then Q has no terms. (I.e. $Q = 0$.) On the other hand $Q = p(01)p(23) - p(02)p(13) + p(03)p(12)$ is a fully simplified quadratic relation even if, for example, $p(01) = 0$.

When $Q = 0$ we say that Q is trivial. When $Q = \pm Q'$ we say that Q and Q' are identical.

We proceed with some observations. Suppose $j_0,...,j_{d-1}$ and $k_0,...,k_{d+1}$ generate the quadratic relation Q. Suppose similarly that $j'_0,...,j'_{d-1}$ and $k'_0,...,k'_{d+1}$ generate the quadratic relation Q'. We have the following.

Lemma 1. *If $j_0,...,j_{d-1}$ are not distinct then Q is trivial.*

Proof.

$$\sum_{\lambda=0}^{d+1} (-1)^\lambda p(j_0...j_{d-1}k_\lambda)p(k_0...\widehat{k_\lambda}...k_{d+1}) = \sum_{\lambda=0}^{d+1} 0 = 0.$$

$\blacksquare$

Lemma 2. *If $k_0,\ldots,k_{d+1}$ are not distinct then Q is trivial.*

Proof. If three or more of the k_β are the same then Q is trivial as above. If $k_\alpha = k_\beta$, where $\alpha < \beta$, then

$$\textstyle\sum_{\lambda=0}^{d+1}(-1)^\lambda p(j_0\ldots j_{d-1}k_\lambda)p(k_0\ldots\widehat{k_\lambda}\ldots k_{d+1}) =$$

$$(-1)^\alpha p(j_0\ldots j_{d-1}k_\alpha)p(k_0\ldots\widehat{k_\alpha}\ldots k_{d+1}) + (-1)^\beta p(j_0\ldots j_{d-1}k_\beta)p(k_0\ldots\widehat{k_\beta}\ldots k_{d+1}) =$$

$$(-1)^\alpha p(j_0\ldots j_{d-1}k_\alpha)[p(k_0\ldots\widehat{k_\alpha}\ldots k_{d+1}) + (-1)^{\beta-\alpha}p(k_0\ldots\widehat{k_\beta}\ldots k_{d+1})] = 0$$

since $p(k_0\ldots\widehat{k_\alpha}\ldots k_{d+1}) = p(k_0\ldots\widehat{k_\beta}\ldots k_{d+1})$ when $\beta-\alpha$ is odd and $p(k_0\ldots\widehat{k_\alpha}\ldots k_{d+1}) = -p(k_0\ldots\widehat{k_\beta}\ldots k_{d+1})$ when $\beta-\alpha$ is even. $\blacksquare$

Remark. By virtue of Lemmas 1 and 2, we will assume from here on that any two generating sequences $\mathbf{j}$ and $\mathbf{k}$ have, respectively, distinct terms.

Lemma 3. *If $J = J'$ with $\mathbf{k} = \mathbf{k}'$ then Q and Q' are identical.*

Proof. In this case $j'_0\ldots j'_{d-1}$ is a rearrangement of $j_0\ldots j_{d-1}$. Thus we have an integer m such that $p(j'_0\ldots j'_{d-1}k_\lambda) = (-1)^m p(j_0\ldots j_{d-1}k_\lambda)$ for the k_λ. Hence

$$\textstyle\sum_{\lambda=0}^{d+1}(-1)^\lambda p(j'_0\ldots j'_{d-1}k_\lambda)p(k_0\ldots\widehat{k_\lambda}\ldots k_{d+1}) =$$

$$(-1)^m\textstyle\sum_{\lambda=0}^{d+1}(-1)^\lambda p(j_0\ldots j_{d-1}k_\lambda)p(k_0\ldots\widehat{k_\lambda}\ldots k_{d+1}).$$

$\blacksquare$

Lemma 4. *If $K = K'$ with $\mathbf{j} = \mathbf{j}'$ then Q and Q' are identical.*

Proof. Here $k'_0\ldots k'_{d+1}$ can be obtained from $k_0\ldots k_{d+1}$ using a finite number of interchanges of the form $k_0\ldots k_{\beta+1}k_\beta\ldots k_{d+1}$, where $0 \le \beta \le d$. It therefore suffices to show that Q remains unaltered (up to sign) by such interchanges. Let $k'_0\ldots k'_{d+1} = k_0\ldots k_{\beta+1}k_\beta\ldots k_{d+1}$. Then

$$p(k'_0\ldots\widehat{k'_\lambda}\ldots k'_{d+1}) = p(k_0\ldots\widehat{k_\lambda}\ldots k_{\beta+1}k_\beta\ldots k_{d+1}) = -p(k_0\ldots\widehat{k_\lambda}\ldots k_{d+1})$$

when $\lambda \ne \beta, \beta+1$ and

$$p(k'_0\ldots\widehat{k'_\lambda}\ldots k'_{d+1}) = p(k_0\ldots\widehat{k_{\beta+1}}k_\beta\ldots k_{d+1}) = p(k_0\ldots\widehat{k_{\beta+1}}\ldots k_{d+1})$$

when $\lambda = \beta$. (Similar when $\lambda = \beta+1$.) Therefore

$$\textstyle\sum_{\lambda=0}^{d+1}(-1)^\lambda p(j_0\ldots j_{d-1}k'_\lambda)p(k'_0\ldots\widehat{k'_\lambda}\ldots k'_{d+1}) =$$

$$\begin{gathered}-\textstyle\sum_{\lambda\ne\beta,\beta+1}(-1)^\lambda p(j_0\ldots j_{d-1}k_\lambda)p(k_0\ldots\widehat{k_\lambda}\ldots k_{d+1}) + \\ (-1)^\beta p(j_0\ldots j_{d-1}k_{\beta+1})p(k_0\ldots\widehat{k_{\beta+1}}\ldots k_{d+1}) + \\ (-1)^{\beta+1}p(j_0\ldots j_{d-1}k_\beta)p(k_0\ldots\widehat{k_\beta}\ldots k_{d+1}) =\end{gathered}$$

$$-\textstyle\sum_{\lambda=0}^{d+1}p(j_0\ldots j_{d-1}k_\lambda)p(k_0\ldots\widehat{k_\lambda}\ldots k_{d+1}).$$

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Lemma 5. *If $J \subseteq K$ then Q is trivial.*

Proof. Using Lemmas 3 and 4, we can write $j_0 ... j_{d-1} = k_0 ... k_{d-1}$. Then we have

$$\sum_{\lambda=0}^{d+1} (-1)^\lambda p(j_0 ... j_{d-1} k_\lambda) p(k_0 ... \widehat{k_\lambda} ... k_{d+1}) =$$

$$(-1)^d p(j_0 ... j_{d-1} k_d) p(k_0 ... k_{d-1} k_{d+1}) + (-1)^{d+1} p(j_0 ... j_{d-1} k_{d+1}) p(k_0 ... k_d) =$$

$$(-1)^d p(k_0 ... k_d) p(k_0 ... k_{d-1} k_{d+1}) - (-1)^d p(k_0 ... k_{d-1} k_{d+1}) p(k_0 ... k_d) = 0.$$

∎

Remark. As with the remark following Lemmas 1 and 2, we will assume from now on, using Lemma 5, that for any two generating sequences $\mathbf{j}$ and $\mathbf{k}$, we have $J \not\subseteq K$.

Lemma 6. *Q has $\#(K - J)$ terms.*

Proof. Let $\mathbf{i} = \mathbf{j} \cap \mathbf{k}$, $\beta = \#I - 1$. Again using Lemmas 3 and 4, we can write $j_0 ... j_\beta = k_0 ... k_\beta = \mathbf{i}$, $j_{\beta+1} ... j_{d-1} = \mathbf{j} - \mathbf{i}$, and $k_{\beta+1} ... k_{d+1} = \mathbf{k} - \mathbf{i}$. Then

$$\sum_{\lambda=0}^{d+1} (-1)^\lambda p(j_0 ... j_{d-1} k_\lambda) p(k_0 ... \widehat{k_\lambda} ... k_{d+1}) =$$

$$\sum_{\lambda=\beta+1}^{d+1} (-1)^\lambda p(\mathbf{i} j_{\beta+1} ... j_{d-1} k_\lambda) p(\mathbf{i} k_{\beta+1} ... \widehat{k_\lambda} ... k_{d+1}) =$$

$$(-1)^{\beta+1} p(\mathbf{i} j_{\beta+1} ... j_{d-1} k_{\beta+1}) p(\mathbf{i} \widehat{k_{\beta+1}} ... k_{d+1}) + \cdots +$$
$$(-1)^{d+1} p(\mathbf{i} j_{\beta+1} ... j_{d-1} k_{d+1}) p(\mathbf{i} k_{\beta+1} ... k_d \widehat{k_{d+1}}).$$

It follows that Q has at most $(d+2) - (\beta+1) = (d+2) - \#I = \#K - \#(J \cap K) = \#(K - J)$ terms.

It remains to show that each term $p(\mathbf{i} j_{\beta+1} ... j_{d-1} k_\lambda) p(\mathbf{i} k_{\beta+1} ... \widehat{k_\lambda} ... k_{d+1})$ is distinct for $(\beta + 1) \le \lambda \le (d + 1)$. We claim that in fact $p(\mathbf{i} j_{\beta+1} ... j_{d-1} k_\lambda)$ occurs only once in Q for each λ. Since $k_{\beta+1}, ..., k_{d+1}$ are all distinct we see that $p(\mathbf{i} j_{\beta+1} ... j_{d-1} k_{\beta+1}), ..., p(\mathbf{i} j_{\beta+1} ... j_{d-1} k_{d+1})$ are all distinct, and because $j_{d-1} \neq k_{\beta+1}, ..., k_{d+1}$ we have that $p(\mathbf{i} j_{\beta+1} ... j_{d-1} k_\lambda) \neq p(\mathbf{i} \widehat{k_{\beta+1}} ... k_{d+1}), ..., p(\mathbf{i} k_{\beta+1} ... \widehat{k_{d+1}})$ for any λ.

The claim follows and we are done. ∎

Remark. A consequence of the previous lemmas, especially Lemma 6, is that any nontrivial quadratic relation Q has three or more terms, each term with coefficient ± 1. A consequence of the defining sum is that Q cannot exceed $(d + 2)$ terms.

We conclude our preliminary results with one last observation.

Lemma 7. *If $\#(K - J) = 3$ with $J \cup K = J' \cup K'$ and $J \cap K = J' \cap K'$ then Q and Q' are identical.*

Proof. Let $\mathbf{i} = \mathbf{j} \cap \mathbf{k}$. By hypothesis, $\#(J - I) = 1$. Thus we can write $j_0...j_{d-2} = j'_0...j'_{d-2} = k_0...k_{d-2} = k'_0...k'_{d-2} = \mathbf{i}$. If $j_{d-1} = j'_{d-1}$ we are done. Otherwise, we can write $j_{d-1} = k'_{d-1}$, $k_{d-1} = j'_{d-1}$, $k_d = k'_d$, and $k_{d+1} = k'_{d+1}$. Now calculate

$$\sum_{\lambda=0}^{d+1} (-1)^\lambda p(j'_0...j'_{d-1}k'_\lambda)p(k'_0...\widehat{k'_\lambda}...k'_{d+1}) =$$

$$\sum_{\lambda=0}^{d+1} (-1)^\lambda p(\mathbf{i}j'_{d-1}k'_\lambda)p(k'_0...\widehat{k'_\lambda}...k'_{d+1}) =$$

$$\begin{gathered}(-1)^{d-1}p(\mathbf{i}j'_{d-1}k'_{d-1})p(\mathbf{i}k'_dk'_{d+1})+\\ (-1)^d p(\mathbf{i}j'_{d-1}k'_d)p(\mathbf{i}k'_{d-1}k'_{d+1})+\\ (-1)^{d+1}p(\mathbf{i}j'_{d-1}k'_{d+1})p(\mathbf{i}k'_{d-1}k'_d) =\end{gathered}$$

$$\begin{gathered}(-1)^{d-1}p(\mathbf{i}k_{d-1}j_{d-1})p(\mathbf{i}k_dk_{d+1})+\\ (-1)^d p(\mathbf{i}k_{d-1}k_d)p(\mathbf{i}j_{d-1}k_{d+1})+\\ (-1)^{d+1}p(\mathbf{i}k_{d-1}k_{d+1})p(\mathbf{i}j_{d-1}k_d) =\end{gathered}$$

$$\begin{gathered}-(-1)^{d-1}p(\mathbf{i}j_{d-1}k_{d-1})p(k_0...\widehat{k_{d-1}}...k_{d+1})+\\ -(-1)^d p(\mathbf{i}j_{d-1}k_d)p(k_0...\widehat{k}_dk_{d+1})+\\ -(-1)^{d+1}p(\mathbf{i}j_{d-1}k_{d+1})p(k_0...k_d\widehat{k_{d+1}}) =\end{gathered}$$

$$-\sum_{\lambda=0}^{d+1} (-1)^\lambda p(j_0...j_{d-1}k_\lambda)p(k_0...\widehat{k_\lambda}...k_{d+1}).$$

∎

3 A Minimal Generating Set for the Quadratic Relations

We are now ready to define a minimal generating set S for the set QR of nontrivial quadratic relations (modulo identical relations).

Holding for $3 \leq m \leq (d+2)$ we define QR_m to be the set of m-term quadratic relations (again modulo identical relations) and we take S_m to be the (possibly empty) set of pairs $(\mathbf{j}, \mathbf{k}) = (\mathbf{i}\widetilde{\mathbf{j}}, \mathbf{i}\widetilde{\mathbf{k}})$ such that $\mathbf{i}, \widetilde{\mathbf{j}}, \widetilde{\mathbf{k}}$ are strictly increasing pairwise disjoint sequences with $\#I = (d - m + 2)$, $\#\widetilde{J} = (m - 2)$ and $\#\widetilde{K} = m$. When $m = 3$ we also require $\widetilde{\mathbf{j}}\widetilde{\mathbf{k}}$ to be strictly increasing, that is we require the term $\widetilde{j}_0$ of $\widetilde{\mathbf{j}}$ to be strictly less than each term of $\widetilde{\mathbf{k}}$.

Then $S = \bigcup_{m=3}^{d+2} S_m$. We will show that S is in bijective correspondence with QR via the map $g : S \to QR$ given by

$$g(\mathbf{j}, \mathbf{k}) = \sum_{\lambda=0}^{d+1} (-1)^\lambda p(j_0...j_{d-1}k_\lambda)p(k_0...\widehat{k_\lambda}...k_{d+1}).$$

Since $\#(K - J) = \#\widetilde{K} \geq 3$ for all $(\mathbf{j}, \mathbf{k}) \in S$ it folllows from Lemma 6 that g is well-defined. It remains to show that g is a bijection.

Lemma 8. *g is onto.*

Proof. Suppose Q is an m-term quadratic relation. We have $(\mathbf{j}', \mathbf{k}')$ such that $g(\mathbf{j}', \mathbf{k}') = Q$. (Note that $(\mathbf{j}', \mathbf{k}')$ is not necessarily in S.)

Let $\mathbf{i}$ be strictly increasing with $I = J' \cap K'$. Using Lemma 6, we see that $\#(K' - J') = m$ so that $\#I = \#(J' \cap K') = (d - m + 2)$. In addition, $\#(J' - K') = d - (d - m + 2) = (m - 2)$.

If $m > 3$ let $\widetilde{\mathbf{j}}, \widetilde{\mathbf{k}}$ be strictly increasing with $\widetilde{J} = (J' - K')$ and $\widetilde{K} = (K' - J')$. Then $(\mathbf{j}, \mathbf{k}) = (\mathbf{i}\widetilde{\mathbf{j}}, \mathbf{i}\widetilde{\mathbf{k}})$ is in S_m and is a rearrangement of $(\mathbf{j}', \mathbf{k}')$. Hence $g(\mathbf{j}, \mathbf{k}) = g(\mathbf{j}', \mathbf{k}') = Q$.

If $m = 3$ let $\widetilde{\mathbf{j}}\widetilde{\mathbf{k}} = \widetilde{j}_0 \widetilde{k}_0 \widetilde{k}_1 \widetilde{k}_2$ be strictly increasing with $\widetilde{J} \cup \widetilde{K} = (J' \cup K') - I$. Then $(\mathbf{j}, \mathbf{k}) = (\mathbf{i}\widetilde{\mathbf{j}}, \mathbf{i}\widetilde{\mathbf{k}})$ is in S_3 and by Lemma 7 we see that $g(\mathbf{j}, \mathbf{k}) = Q$. ∎

Lemma 9. *g is one-to-one.*

Proof. Take $(\mathbf{j}, \mathbf{k}) = (\mathbf{i}\widetilde{\mathbf{j}}, \mathbf{i}\widetilde{\mathbf{k}})$ and $(\mathbf{j}', \mathbf{k}') = (\mathbf{i}'\widetilde{\mathbf{j}}', \mathbf{i}'\widetilde{\mathbf{k}}')$ in S with $(\mathbf{j}, \mathbf{k}) \neq (\mathbf{j}', \mathbf{k}')$. We want to show that $g(\mathbf{j}, \mathbf{k}) \neq g(\mathbf{j}', \mathbf{k}')$.

If $(\mathbf{j}, \mathbf{k}) \in S_m$ and $(\mathbf{j}', \mathbf{k}') \in S_q$ with $m \neq q$ then $g(\mathbf{j}, \mathbf{k})$ has m terms while $g(\mathbf{j}', \mathbf{k}')$ has q terms. Whence $g(\mathbf{j}, \mathbf{k}) \neq g(\mathbf{j}', \mathbf{k}')$ and we are done. Thus assume both $(\mathbf{j}, \mathbf{k})$ and $(\mathbf{j}', \mathbf{k}')$ are in S_m.

Define $\mathbf{h} = \mathbf{j} \cup \mathbf{k}$ and $\mathbf{h}' = \mathbf{j}' \cup \mathbf{k}'$. Since $\mathbf{h} = \mathbf{h}'$, $\mathbf{i} = \mathbf{i}'$ and $\widetilde{\mathbf{j}} = \widetilde{\mathbf{j}}'$ implies $(\mathbf{j}, \mathbf{k}) = (\mathbf{j}', \mathbf{k}')$ there are three cases.

Case I. $\mathbf{h} \neq \mathbf{h}'$. Then we have $h_\beta \in H$ with $h_\beta \notin H'$. In this case h_β occurs in each term of $g(\mathbf{j}, \mathbf{k})$ but in no term of $g(\mathbf{j}', \mathbf{k}')$: if $h_\beta \in J$ then h_β occurs in $p(j_0 ... j_{d-1} k_\lambda)$ of $p(j_0 ... j_{d-1} k_\lambda) p(k_0 ... \widehat{k_\lambda} ... k_{d+1})$ for each λ; if $h_\beta \in K - J$ then h_β occurs in $p(j_0 ... j_{d-1} k_\lambda)$ of $p(j_0 ... j_{d-1} k_\lambda) p(k_0 ... \widehat{k_\lambda} ... k_{d+1})$ when $k_\lambda = h_\beta$ and h_β occurs in $p(k_0 ... \widehat{k_\lambda} ... k_{d+1})$ of $p(j_0 ... j_{d-1} k_\lambda) p(k_0 ... \widehat{k_\lambda} ... k_{d+1})$ when $k_\lambda \neq h_\beta$.

Case II. $\mathbf{i} \neq \mathbf{i}'$. By Case I we assume $\mathbf{h} = \mathbf{h}'$. Since $0 \leq \lambda < (d - m + 2)$ gives trivial terms of $g(\mathbf{j}, \mathbf{k}), g(\mathbf{j}', \mathbf{k}')$ we also assume $(d - m + 2) \leq \lambda \leq (d + 1)$.

We have $i_\beta \in I$ but $i_\beta \notin I'$. We claim that i_β occurs in both $p(\mathbf{i} j_{d-m+2} ... j_{d-1} k_\lambda)$ and $p(\mathbf{i} k_{d-m+2} ... \widehat{k_\lambda} ... k_{d+1})$ of $p(\mathbf{i} j_{d-m+2} ... j_{d-1} k_\lambda) p(\mathbf{i} k_{d-m+2} ... \widehat{k_\lambda} ... k_{d+1})$ for each λ but in only one of $p(\mathbf{i}' j'_{d-m+2} ... j'_{d-1} k'_\lambda)$ and $p(\mathbf{i}' k'_{d-m+2} ... \widehat{k'_\lambda} ... k'_{d+1})$: if $i_\beta \in J' - K' = \widetilde{J}'$ then i_β occurs in $p(\mathbf{i}' j'_{d-m+2} ... j'_{d-1} k'_\lambda)$ but not in $p(\mathbf{i}' k'_{d-m+2} ... \widehat{k'_\lambda} ... k'_{d+1})$ for all λ; otherwise, $i_\beta \in K' - J' = \widetilde{K'}$ so that i_β occurs in either $p(\mathbf{i}' j'_{d-m+2} ... j'_{d-1} k'_\lambda)$ or $p(\mathbf{i}' k'_{d-m+2} ... \widehat{k_\lambda} ... k'_{d+1})$ but not in both.

We note that if $m = 3$ we are done since $\mathbf{h} = \mathbf{h}'$ and $\mathbf{i} = \mathbf{i}'$ implies $\widetilde{\mathbf{j}} = \widetilde{\mathbf{j}}'$ when $(\mathbf{j}, \mathbf{k}), (\mathbf{j}', \mathbf{k}')$ are in S_3. Otherwise, we continue.

Case III. $\widetilde{\mathbf{j}} \neq \widetilde{\mathbf{j}}'$. (Assume $\mathbf{h} = \mathbf{h}'$ and $\mathbf{i} = \mathbf{i}'$.) We will show that a factor $p(j_0 ... j_{d-1} k_\lambda)$ of a term $p(j_0 ... j_{d-1} k_\lambda) p(k_0 ... \widehat{k_\lambda} ... k_{d+1})$ of $g(\mathbf{j}, \mathbf{k})$ never occurs in $g(\mathbf{j}', \mathbf{k}')$.

Since the terms of $g(\mathbf{j}, \mathbf{k})$ and $g(\mathbf{j}', \mathbf{k}')$ are unaltered by rearrangements of $(\mathbf{j}, \mathbf{k}) = (\mathbf{i}\widetilde{\mathbf{j}}, \mathbf{i}\widetilde{\mathbf{k}})$ and $(\mathbf{j}', \mathbf{k}') = (\mathbf{i}'\widetilde{\mathbf{j}}', \mathbf{i}'\widetilde{\mathbf{k}}')$, we can assume that $\widetilde{\mathbf{j}} = j_{d-m+2} ... j_{d-1}$, $\widetilde{\mathbf{k}} = k_{d-m+2} ... k_{d+1}$ and $\widetilde{\mathbf{j}}' = j'_{d-m+2} ... j'_{d-1}$, $\widetilde{\mathbf{k}}' = k'_{d-m+2} ... k'_{d+1}$ are ordered as follows:

$$\begin{aligned}
&j_{d-m+2}...j_{\alpha-1} = k'_{d-m+2}...k'_{\alpha-1} = (\mathbf{j}\cap\mathbf{k}') - \mathbf{i}\\
&j_{\alpha}...j_{d-1} = j'_{\alpha}...j'_{d-1} = (\mathbf{j}\cap\mathbf{j}') - \mathbf{i}\\
&k_{d-m+2}...k_{\alpha-1} = j'_{d-m+2}...j'_{\alpha-1} = (\mathbf{k}\cap\mathbf{j}') - \mathbf{i}\\
&k_{\alpha}...k_{d+1} = k'_{\alpha}...k'_{d+1} = (\mathbf{k}\cap\mathbf{k}') - \mathbf{i}
\end{aligned}$$

We note that $(d-m+2) < \alpha \leq d$.

Thus we get

$$g(\mathbf{j},\mathbf{k}) = \textstyle\sum_{\lambda=0}^{d+1}(-1)^{\lambda}p(\mathbf{i}j_{d-m+2}...j_{d-1}k_{\lambda})p(k_0...\widehat{k_{\lambda}}...k_{d+1})$$

$$= \textstyle\sum_{\lambda=d-m+2}^{d+1}(-1)^{\lambda}p(\mathbf{i}j_{d-m+2}...j_{d-1}k_{\lambda})p(\mathbf{i}k_{d-m+2}...\widehat{k_{\lambda}}...k_{d+1})$$

$$\begin{aligned}
= &\textstyle\sum_{\lambda=d-m+2}^{\alpha-1}(-1)^{\lambda}p(\mathbf{i}j_{d-m+2}...j_{d-1}k_{\lambda})p(\mathbf{i}k_{d-m+2}...\widehat{k_{\lambda}}...k_{\alpha-1}k_{\alpha}...k_{d+1})\\
&+\textstyle\sum_{\lambda=\alpha}^{d+1}(-1)^{\lambda}p(\mathbf{i}j_{d-m+2}...j_{d-1}k_{\lambda})p(\mathbf{i}k_{d-m+2}...k_{\alpha-1}k_{\alpha}...\widehat{k_{\lambda}}...k_{d+1})
\end{aligned}$$

$$\begin{aligned}
= &\textstyle\sum_{\lambda=d-m+2}^{\alpha-1}(-1)^{\lambda}p(\mathbf{i}k'_{d-m+2}...k'_{\alpha-1}j'_{\alpha}...j'_{d-1}j'_{\lambda})p(\mathbf{i}j'_{d-m+2}...\widehat{j'_{\lambda}}...j'_{\alpha-1}k'_{\alpha}...k'_{d+1})\\
&+\textstyle\sum_{\lambda=\alpha}^{d+1}(-1)^{\lambda}p(\mathbf{i}k'_{d-m+2}...k'_{\alpha-1}j'_{\alpha}...j'_{d-1}k'_{\lambda})p(\mathbf{i}j'_{d-m+2}...j'_{\alpha-1}k'_{\alpha}...\widehat{k'_{\lambda}}...k'_{d+1}).
\end{aligned}$$

Since

$$g(\mathbf{j}',\mathbf{k}') = \textstyle\sum_{\lambda=0}^{d+1}(-1)^{\lambda}p(\mathbf{i}j'_{d-m+2}...j'_{d-1}k'_{\lambda})p(k'_0...\widehat{k'_{\lambda}}...k_{d+1})$$

$$= \textstyle\sum_{\lambda=d-m+2}^{d+1}(-1)^{\lambda}p(\mathbf{i}j'_{d-m+2}...j'_{d-1}k'_{\lambda})p(\mathbf{i}k'_{d-m+2}...\widehat{k'_{\lambda}}...k'_{d+1})$$

there are two possibilites.

If $\alpha < d$ then any factor $p(\mathbf{i}k'_{d-m+2}...k'_{\alpha-1}j'_{\alpha}...j'_{d-1}k'_{\lambda})$ from the second sum in $g(\mathbf{j},\mathbf{k})$ contains at least two (distinct) terms k'_{β}, k'_{γ} of $k'_{d-m+2}...k'_{d+1} = \widetilde{\mathbf{k}'}$ and at least one term j'_{δ} of $j'_{d-m+2}...j'_{d-1} = \widetilde{\mathbf{j}'}$. I.e. there is a factor in $g(\mathbf{j},\mathbf{k})$ which never occurs in $g(\mathbf{j}',\mathbf{k}')$.

If $\alpha = d$ then one of the factors $p(\mathbf{i}k'_{d-m+2}...k'_{d-1}j'_{\lambda})$ from the first sum in $g(\mathbf{j},\mathbf{k})$ contains at least two terms k'_{β}, k'_{γ} of $k'_{d-m+2}...k'_{d+1} = \widetilde{\mathbf{k}'}$ and at least one term j'_{δ} of $j'_{d-m+2}...j'_{d-1} = \widetilde{\mathbf{j}'}$. ∎

Lemmas 8 and 9 yield the following main result.

Theorem 10. *S is minimal generating set for QR. That is, $g : S \rightarrow QR$ is a bijection. And in fact more is true: for $3 \leq m \leq (d+2)$ we have that $g|_{S_m} : S_m \rightarrow QR_m$ is bijective so that S_m is a minimal generating set for QR_m.*

As an application of Theorem 10, we count the quadratic relations.

Corollary 11. $\#QR =$

$$\binom{n+1}{d+3}\binom{d+3}{d-1} + \sum_{m=4}^{d+2}\binom{n+1}{d+m}\binom{d+m}{d-m+2}\binom{2m-2}{m},$$

where $\begin{pmatrix} q \\ r \end{pmatrix} = 0$ *if* $r > q$.

Proof. If for each S_m we pick $\mathbf{i} \cup \widetilde{\mathbf{j}} \cup \widetilde{\mathbf{k}} = \mathbf{j} \cup \mathbf{k}$ first, $\mathbf{i} = \mathbf{j} \cap \mathbf{k}$ next and $\widetilde{\mathbf{j}}$ last (so that $\widetilde{\mathbf{k}} = \mathbf{j} \cup \mathbf{k} - (\mathbf{i} \cup \widetilde{\mathbf{j}})$), we have

$$\#S_3 = \binom{n+1}{d+3} \binom{d+3}{d-1}$$

and

$$\#S_m = \binom{n+1}{d+m} \binom{d+m}{d-m+2} \binom{2m-2}{m}$$

for $m > 3$. Thus,

$$\#QR = \#S = \#S_3 + \sum_{m=4}^{d+2} \#S_m.$$

∎

4 The Linear Independence of the Quadratic Relations

In general, the quadratic relations are not linearly independent. The following example provides an illustration. (Recall that we denote the Grassman Manifold of d-planes in projective n-space by $G_{d,n}$.)

Example 12. *Linear dependence among the* 4*-term quadratic relations defining* $G_{2,5}$*:*

$$
\begin{aligned}
&[p(012)p(345) - p(013)p(245) + p(014)p(235) - p(015)p(234)] \\
&-[-p(012)p(345) - p(023)p(145) + p(024)p(135) - p(025)p(134)] \\
&+[-p(013)p(245) + p(023)p(145) + p(034)p(125) - p(035)p(124)] \\
&+[-p(014)p(235) + p(024)p(135) - p(034)p(125) - p(045)p(123)] \\
&-[-p(015)p(234) + p(025)p(134) - p(035)p(124) + p(045)p(123)] \\
&+2[p(045)p(123) - p(145)p(023) + p(245)p(013) - p(345)p(012)] = 0.
\end{aligned}
$$

∎

There are, of course, certain subsets of the quadratic relations which are linearly independent.

Lemma 13. QR_3 *forms a linearly independent set.*

Proof. We show in fact that every term in each quadratic relation $Q \in QR_3$ is distinct. The claim follows.

Take $Q \in QR_3$ generated uniquely (using Theorem 10) by $(\mathbf{j}, \mathbf{k}) \in S_3$. Choose a term $T = p(j_0 \ldots j_{d-1} k_\alpha) p(k_0 \ldots \widehat{k_\alpha} \ldots k_{d+1})$ of Q. Consider any *other* quadratic relation $P \in QR_3$ generated uniquely by $(\mathbf{j}', \mathbf{k}') \in S_3$ and choose a term $V = p(j_0' \ldots j_{d-1}' k_\beta') p(k_0' \ldots \widehat{k_\beta'} \ldots k_{d+1}')$ of P. Since P and Q are not identical ($P \neq \pm Q$) we see that $(\mathbf{j}, \mathbf{k}) \neq (\mathbf{j}', \mathbf{k}')$. Thus $T \neq V$ using Cases I and II of Lemma 9 and we are done. ∎

We also have the following "near" linear independence among the quadratic relations.

Lemma 14. *An m-term quadratic relation Q cannot be expressed as a linear combination of q-term relations when $m \neq q$.*

Proof. Take an m-term quadratic relation Q generated uniquely by $(\mathbf{j}, \mathbf{k}) \in S_m$. Similarly take any q-term relation P generated uniquely by $(\mathbf{j}', \mathbf{k}') \in S_q$.

Let $\mathbf{h} = \mathbf{j} \cup \mathbf{k}$ and $\mathbf{h}' = \mathbf{j}' \cup \mathbf{k}'$. Then $\#H = (d+m)$ and $\#H' = (d+q)$. Since $m \neq q$ we see that $\#H \neq \#H'$ so that $\mathbf{h} \neq \mathbf{h}'$. Using Case I of Lemma 9 we see that no term of Q appears in P. ∎

As a special case of Lemma 14 we have one last notable result.

Theorem 15. *No quadratic relation with more than three terms can be expressed as a linear combination of three-term relations.*

And finally, in consideration of the above, we have

Conjecture 16.

$$\dim(\text{span } QR) = \#QR_3 + \sum_{m=4}^{d+2} \binom{n+1}{d+m} \binom{d+m}{d-m+2} (2m-3).$$

∎

5 Concluding Examples

We conclude our discussion of the quadratic relations which define the Grassman Manifold via two minor applications of the results from Sections 3 and 4.

Applying Theorem 10 we see that

Example 17. *The quadratic relations*

$$p(012)p(034) - p(013)p(024) + p(014)p(023) = 0$$

$$-p(012)p(134) + p(013)p(124) - p(014)p(123) = 0$$

$$p(012)p(234) - p(023)p(124) + p(024)p(123) = 0$$

$$-p(013)p(234) + p(023)p(134) - p(034)p(123) = 0$$

$$p(014)p(234) - p(024)p(134) + p(034)p(124) = 0$$

completely define $G_{2,4}$.

Applying Lemma 13 we have

Example 18. *The quadratic relations defining $G_{1,n}$ are linearly independent for all n. This is true because the quadratic relations defining $G_{1,n}$ are all three-term relations. (This also holds for $G_{n-2,n}$.)*

6 Bibliography

[Kl&La72] Kleiman, S. and Laksov, D., "Schubert Calculus," American Mathematical Monthly Vol. 79 #10 (1972) pg. 1061-1081.