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Hexanions: 6D Space for Twists

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6D Space for Twists.

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Abstract

Some of the popular representations for the motion of a rigid object are 4×4 real matrices or combinations of vector and quaternion [Ale02][DKL98]. A drawback of using 4×4 real matrices is their over-general purpose. A real matrix is capable of representing a far too rich set of transformations, for instance scaling and shear. Another drawback with matrices is the accumulation of numerical errors, which impose to renormalize the matrix once in a while. On the other hand, a drawback of using a combination of vector and quaternion is the separation of the motion in two components that have to be handled individually. This component separation can make implementation on computing devices unclear or unnecessary long. While there are surely other alternatives for representing numerically rigid motion, we propose here a new representation called *hexanion*. A hexanion is a compact and natural representation of the rigid transformations: translations, rotations and twists. The number of dimensions of hexanion space is six, which is also the degrees of freedom of rigid transformations.

1 Motivation

The motivation behind the formalism of hexanions is the necessity to compute fractions of rigid transformation. For a transformation represented by a 4×4 real matrix, taking a fraction h can be done with the following:

$$M^h = \exp(h \log M) \tag{1}$$

$$\text{where} \quad \begin{array}{rcl} \exp M &=& \sum_{k=0}^{\infty} \frac{M^k}{k!} \\ \log M &=& -\sum_{k=1}^{\infty} \frac{(I-M)^k}{k} \end{array} \tag{2}$$

M. Alexa et al. [Ale02] proposes to use numerical methods to achieve this in the general case. A matrix is however a poor representation for rigid transformations. A better choice is to use a hexanions, which can be interpreted as a shorthand for $\log M$ when M is a twist. A hexanion is a 6d element denoted with a pair of vectors:

$$\langle \vec{\omega}, \vec{m} \rangle \tag{3}$$

The relation beween a hexanion and the logarithm of a twist is:

$$\log M = \begin{pmatrix} 0 & -\omega_z & \omega_y & m_x \\ \omega_z & 0 & -\omega_x & m_y \\ -\omega_y & \omega_x & 0 & m_z \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

The above is called the hexanion's matrix form, and is denoted with round brackets, $(\vec{\omega}, \vec{m})$. Note that in a hexanion, the part $\vec{\omega}$ is in fact the logarithm of a quaternion.

2 Overview

The hexanion space is a *real vector space*, whose elements represent twists, also known as screws. A twist is a translation of magnitude t in a direction along a line defined by a point c and a vector $\vec{n}$ together with a rotation of angle θ about axis $\vec{n}$. Mozzi and Cauchy have proved that any motion of a rigid body in space at every instant is a twist motion [Wei04]. The hexanion of the twist transformation defined as above is:

$$\langle \theta\vec{n}, \theta c \times \vec{n} + t\vec{n} \rangle \tag{5}$$

Pure translation or rotation are obtained by setting θ or t to zero. Combining or interpolating twists with hexanion is as simple as vector algebra. For instance if $\langle \vec{\omega}, \vec{m} \rangle$ denotes the new position, the motion is described as follows:

$$\langle u\vec{\omega}, u\vec{m} \rangle, \quad u \in [0,1] \tag{6}$$

With the definition of operator $*$ described below, the trajectory parameterized in u of the point of a rigid object is described as follows:

$$\langle u\vec{\omega}, u\vec{m} \rangle * p, \quad u \in [0,1] \tag{7}$$

3 Algebraic properties

In this section, algebraic properties of the hexanion are reviewed.

Algebraic structure: Let us define the hexanion space, a *real vector space* $(\mathbb{R}^6, +, \cdot)$, where:

$$(\mathbb{R}^6, +) \text{ is a commutative group} \tag{8}$$
$$\forall(\lambda, \mu, A) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^6,\ (\lambda + \mu) \cdot A = (\lambda \cdot A) + (\mu \cdot A) \tag{9}$$
$$\forall(\lambda, A, B) \in \mathbb{R} \times \mathbb{R}^6 \times \mathbb{R}^6,\ \lambda \cdot (A + B) = (\lambda \cdot A) + (\lambda \cdot B) \tag{10}$$
$$\forall(\lambda, \mu, A) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^6,\ \lambda \cdot (\mu \cdot A) = (\lambda \cdot \mu) \cdot A \tag{11}$$
$$\forall A \in \mathbb{R}^6, 1 \cdot A = A \tag{12}$$

A pair of 3D vectors is a convenient notation for manipulating elements of hexanion space, denoted with angle brackets:

$$\langle \vec{\omega}, \vec{m} \rangle$$

Let us define two hexanions $A = \langle \vec{\omega}_A, \vec{m}_A \rangle$ and $B = \langle \vec{\omega}_B, \vec{m}_B \rangle$. Loosely speaking, $\lambda \cdot A$ is the hexanion whose components are those of A multiplied by λ, i.e. $\langle \lambda\vec{\omega}_A, \lambda\vec{m}_A \rangle$ and $A + B$ is the hexanion whose components are the sum of respective components in A and B, i.e. $\langle \vec{\omega}_A + \vec{\omega}_B, \vec{m}_A + \vec{m}_B \rangle$.

Notation: A point, p, is represented with homogeneous coordinates $(p_x, p_y, p_z, 1)^\top$. A vector is marked with an arrow, $\vec{v}$, and is represented with homogeneous coordinates $(v_x, v_y, v_z, 0)^\top$. Binary operators $\cdot$ and is defined for points and vectors and denote the dot product. Binary operators $\times$ is defined for points and vectors and denote the cross products, which resutls in a vector. The binary operator $*$ denotes the hexanion-point and hexanion-vector products. We omit the operator for the following products: matrix-matrix, matrix-point, matrix-vector, scalar-matrix scalar-vector, scalar-point and scalar-scalar.

Exponential: In its matrix form, a hexanion can be raised to integer powers like 4×4 real matrices. The integer powers of hexanions matrices are used for defining the exponential of a hexanion matrix, as follows:

$$\exp(\vec{\omega}, \vec{m}) = \sum_{k=0}^{\infty} \frac{(\vec{\omega}, \vec{m})^k}{k!} \tag{13}$$

The exponential of a hexanion is the $M_{4,4}(\mathbb{R})$ projective matrix of a rigid transformation. Conveniently, the exponential of a hexanion has a closed-form, prooved in Appendix 7:

$$\exp(\vec{\omega},\vec{m}) = \begin{cases} I+(\vec{\omega},\vec{m}) & \text{if } \|\vec{\omega}\|=0 \\ I+\frac{1-\cos\|\vec{\omega}\|}{\|\vec{\omega}\|^2}(\vec{\omega},\vec{m})^2+\frac{\sin\|\vec{\omega}\|}{\|\vec{\omega}\|}(\vec{\omega},\vec{m}) & \text{if } \vec{\omega}\cdot\vec{m}=0 \\ I+(\vec{\omega},\vec{m})+\frac{1-\cos\|\vec{\omega}\|}{\|\vec{\omega}\|^2}(\vec{\omega},\vec{m})^2+\frac{\|\vec{\omega}\|-\sin\|\vec{\omega}\|}{\|\vec{\omega}\|^3}(\vec{\omega},\vec{m})^3 & \text{otherwise} \end{cases} \quad (14)$$

Note that the above is not a piecewise definition of the exponential, but it is a convenient formulation of the cases where the exponential simplifies, since we aim at numerical applications. In some application it can be useful to use a first order approximation of the exponential:

$$\exp(\vec{\omega},\vec{m}) \approx I+(\vec{\omega},\vec{m}) \quad (15)$$

Hexanion-point multiplication: We define the multiplication operator, $*$, of a point, $p=(p_x,p_y,p_z,1)^\top$, by a hexanion $\langle\vec{\omega},\vec{m}\rangle$, as follows:

$$\langle\vec{\omega},\vec{m}\rangle * p = \exp(\vec{\omega},\vec{m})p \quad (16)$$

By using the approximation of the exponential, the above can be approximated if necessary:

$$\langle\vec{\omega},\vec{m}\rangle * p \approx p+\vec{\omega}\times p+\vec{m} \quad (17)$$

Hexanion-normal multiplication: We define the multiplication operator, $*$, of a vector, $\vec{v}=(v_x,v_y,v_z,0)^\top$, by a hexanion $\langle\vec{\omega},\vec{m}\rangle$, as follows:

$$\langle\vec{\omega},\vec{m}\rangle * \vec{v} = \exp(\vec{\omega},\vec{m})\vec{v} \quad (18)$$

By using the approximation of the exponential, the above can be approximated. If the reader decides to use the approximation, he must realize that it does not preserve the length of $\vec{v}$, as opposed to the above closed-form:

$$\langle\vec{\omega},\vec{m}\rangle * \vec{v} \approx \vec{v}+\vec{\omega}\times\vec{v} \quad (19)$$

Neutral: The neutral hexanion element is $\langle 0,0\rangle$. It also satisfies:

$$\langle 0,0\rangle * p = p \quad (20)$$

$$\langle 0,0\rangle * \vec{n} = \vec{n} \quad (21)$$

Hexanion inverse The inverse of $\langle\vec{\omega},\vec{m}\rangle$ is $\langle-\vec{\omega},-\vec{m}\rangle$. The inverse describes the backward motion. The inverse satisfies:

$$\langle-\vec{\omega},-\vec{m}\rangle * (\langle\vec{\omega},\vec{m}\rangle * p) = p \quad (22)$$

4 Hexanion subsets

This section presents the hexanion-to-matrix conversion, the hexanion-point multiplication and the hexanion-vector multiplication. The hexanion-to-matrix conversion is also refered to as the exponential. Hexanions can be classified in three sets: translation, rotation and twists. Knowing the type of an hexanion can simplify computations. In the general case, fast expressions can be derived for particular usage of hexanions.

4.1 Translation-hexanion:

The set of translations is a subset the hexanion space. The hexanion of a translation of vector $\vec{m}$ is:

$$\langle 0, \vec{m} \rangle \tag{23}$$

Conversely, a hexanion $\langle \vec{\omega}, \vec{m} \rangle$ is a translation if $\|\vec{\omega}\| = 0$. The exponential of a translation-hexanion is much simpler than in the general case, and is defined as follows:

$$\exp(0, \vec{m}) = \begin{pmatrix} 1 & 0 & 0 & m_x \\ 0 & 1 & 0 & m_y \\ 0 & 0 & 1 & m_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{24}$$

The multiplication of a point, p, and a normal $\vec{v}$ by a translation-hexanion, $\langle 0, \vec{m} \rangle$, have simple expressions:

$$\begin{array}{rcl} \langle 0, \vec{m} \rangle * p & = & p + \vec{m} \\ \langle 0, \vec{m} \rangle * \vec{v} & = & \vec{v} \end{array} \tag{25}$$

4.2 Rotation-hexanion:

The set of rotations is a subset of the hexanion space. Note that we refer to the set of rotations about a point, more general than unit quaternions which are rotations about the origin. The hexanion of a rotation defined by axis $\vec{\omega}$, center c, and angle $\|\vec{\omega}\|$ is:

$$\langle \vec{\omega}, c \times \vec{\omega} \rangle. \tag{26}$$

Conversely an element $\langle \vec{\omega}, \vec{m} \rangle$ is a rotation if $\vec{\omega} \cdot \vec{m} = 0$. The center, angle and unit axis of rotation are given by the following equations:

$$\begin{array}{rcl} c & = & \frac{\vec{\omega} \times \vec{m}}{\|\vec{\omega}\|^2} \\ \theta & = & \|\vec{\omega}\| \\ \vec{n} & = & \frac{\vec{\omega}}{\|\vec{\omega}\|} \end{array} \tag{27}$$

The exponential of a rotation-hexanion is simpler than in the general case, and is defined as follows:

$$\exp(\vec{\omega}, \vec{m}) = I + \frac{1 - \cos \|\vec{\omega}\|}{\|\vec{\omega}\|^2} (\vec{\omega}, \vec{m})^2 + \frac{\sin \|\vec{\omega}\|}{\|\vec{\omega}\|} (\vec{\omega}, \vec{m}) \tag{28}$$

The multiplication of a point, p, by a rotation $\langle \vec{\omega}, \vec{m} \rangle$, has a closed-form expression:

$$\begin{array}{rcl} \langle \vec{\omega}, \vec{m} \rangle * p & = & p + \frac{1 - \cos \|\vec{\omega}\|}{\|\vec{\omega}\|^2} \vec{\omega} \times \vec{\xi} + \frac{\sin \|\vec{\omega}\|}{\|\vec{\omega}\|} \vec{\xi} \\ \text{where } \vec{\xi} & = & \vec{\omega} \times p + \vec{m} \end{array} \tag{29}$$

The multiplication of a vector, $\vec{v}$, by a rotation $\langle \vec{\omega}, \vec{m} \rangle$, has a closed-form expression:

$$\langle \vec{\omega}, \vec{m} \rangle * \vec{v} \quad = \quad \vec{v} + \frac{1 - \cos \|\vec{\omega}\|}{\|\vec{\omega}\|^2} \vec{\omega} \times (\vec{\omega} \times \vec{v}) + \frac{\sin \|\vec{\omega}\|}{\|\vec{\omega}\|} \vec{\omega} \times \vec{v} \tag{30}$$

4.3 Twist-hexanion:

If we consider translations and rotations to be particular cases of twists, then hexanion space is in fact the set of twists. The hexanion of the twist defined by translation of magnitude t, rotation unit axis $\vec{n}$, center c, and angle θ is:

$$\langle \theta \vec{n}, \theta c \times \vec{n} + \vec{t} \rangle \tag{31}$$

Conversely, if we denote twists with the general 6-d vector $\langle\vec{\omega},\vec{m}\rangle$, the rotation part and the translation part can be identified. We denote these parts Rot and Trs:

$$\langle\vec{\omega},\vec{m}\rangle = Rot\langle\vec{\omega},\vec{m}\rangle + Trs\langle\vec{\omega},\vec{m}\rangle \tag{32}$$

$$\begin{array}{rcl} Rot\langle\vec{\omega},\vec{m}\rangle & = & \langle\vec{\omega}, \frac{1}{\|\vec{\omega}\|^2}\vec{\omega}\times\vec{m}\times\vec{\omega}\rangle \\ Trs\langle\vec{\omega},\vec{m}\rangle & = & \langle 0, \frac{\vec{m}\cdot\vec{\omega}}{\|\vec{\omega}\|^2}\vec{\omega}\rangle \end{array} \tag{33}$$

The exponential of a rotation-hexanion is given by the general case, as follows:

$$\exp(\vec{\omega},\vec{m}) = I + (\vec{\omega},\vec{m}) + \frac{1-\cos\|\vec{\omega}\|}{\|\vec{\omega}\|^2}(\vec{\omega},\vec{m})^2 + \frac{\|\vec{\omega}\|-\sin\|\vec{\omega}\|}{\|\vec{\omega}\|^3}(\vec{\omega},\vec{m})^3 \tag{34}$$

The multiplication of a point, p, by a twist-hexanion $\langle\vec{\omega},\vec{m}\rangle$, has a closed-form expression:

$$\begin{array}{rcl} \langle\vec{\omega},\vec{m}\rangle * p & = & p + \vec{m} + \vec{\omega}\times(\vec{\omega}\times(ap + \frac{1-b}{\|\vec{\omega}\|^2}\vec{m}) + a\vec{m} + bp) \\ \text{where } a & = & \frac{1-\cos(\|\vec{\omega}\|)}{\|\vec{\omega}\|^2} \\ b & = & \frac{\sin(\|\vec{\omega}\|)}{\|\vec{\omega}\|} \end{array} \tag{35}$$

The multiplication of a vector, $\vec{n}$, by a twist-hexanion $\langle\vec{\omega},\vec{m}\rangle$, has a closed-form expression. Notice that this expression is the same as in the case of the rotation-hexanion:

$$\langle\vec{\omega},\vec{m}\rangle * \vec{v} = \vec{v} + \frac{1-\cos\|\vec{\omega}\|}{\|\vec{\omega}\|^2}\vec{\omega}\times(\vec{\omega}\times\vec{v}) + \frac{\sin\|\vec{\omega}\|}{\|\vec{\omega}\|}\vec{\omega}\times\vec{v} \tag{36}$$

5 Properties

The cube of a hexanion matrix is a rotation-hexanion matrix:

$$(\vec{\omega},\vec{m})^3 = (-\|\omega\|^2\vec{\omega}, -\vec{\omega}\times\vec{m}\times\vec{\omega}) \tag{37}$$

The cube of a hexanion is related to the rotation component of a hexanion through the following:

$$Rot(\vec{\omega},\vec{m}) = \frac{-1}{\|\omega\|^2}(\vec{\omega},\vec{m})^3 \tag{38}$$

The square of a hexanion matrix is not a hexanion matrix, but satisfies the following:

$$Rot(\vec{\omega},\vec{m})^2 = (\vec{\omega},\vec{m})^2 \tag{39}$$

In general, the operator $*$ is non-commutative

$$\langle\vec{\omega}_0,\vec{m}_0\rangle * \langle\vec{\omega}_1,\vec{m}_1\rangle * p \neq \langle\vec{\omega}_1,\vec{m}_1\rangle * \langle\vec{\omega}_0,\vec{m}_0\rangle * p \tag{40}$$

The following is a relation between the exponential of a hexanion and the matrix form of that hexanion :

$$\left.\frac{\partial}{\partial h}M^h\right|_{h=0} = (\vec{\omega},\vec{m}) \tag{41}$$

The multiplication of a point or a vector by the matrix form have simple expressions. The following expression involve a matrix-point multiplication, and are useful for developing the closed-form formulas for transforming a point or a normal. They should not be mis-interpreted by the reader as the transform of a point or a normal:

$$\begin{array}{rcl} (\vec{\omega},\vec{m})p & = & \vec{\omega}\times p + \vec{m} \\ (\vec{\omega},\vec{m})\vec{v} & = & \vec{\omega}\times\vec{v} \end{array} \tag{42}$$

Note that the multiplication of a point is a normal, thus $(\vec{\omega},\vec{m})^2 p = \vec{\omega}\times(\vec{\omega}\times p + \vec{m})$.

6 Matrix to hexanion conversion:

This conversion is not meant to be optimal, and is expected to be done only once, for importing a rigid transformation. In a numerical context, we strongly recommend to represented rigid transformations with hexanions and avoid this conversion. Let us define three unit vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$, forming a right-handed orthogonal system. Let us define a position o. These define a 4×4 matrix of a rigid transformation from the origin to that position.

$$M = \begin{pmatrix} x_x & y_x & y_x & o_x \\ x_y & y_y & y_y & o_y \\ x_z & y_z & y_z & o_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{43}$$

We know that this matrix can be written in the form of a hexanion:

$$\langle \theta\vec{n}, \theta\vec{c} \times \vec{n} + t\vec{n} \rangle \tag{44}$$

Vector $\vec{n}$ is the unit eigen-vector of matrix M associated with eigen-value 1:

$$M\vec{n} = \vec{n} \tag{45}$$

The solution for $\vec{n}$ is

$$\vec{n} = \frac{\vec{n}_0}{\|\vec{n}_0\|} \tag{46}$$

$$\text{where } \vec{n}_0 = \begin{cases} \vec{x} + (1 - y_y - z_z, y_x, z_x) & \text{if } 1 + x_x - y_y - z_z \neq 0 \\ \vec{y} + (x_y, 1 - x_x - z_z, z_y) & \text{if } 1 - x_x + y_y - z_z \neq 0 \\ \vec{z} + (x_z, y_z, 1 - x_x - y_y) & \text{if } 1 - x_x - y_y + z_z \neq 0 \end{cases} \tag{47}$$

Once unit vector $\vec{n}$ is found, point c and translation are found by solving the following:

$$Mc = c + t\vec{n} \tag{48}$$

The solution for the magnitude of the translation is:

$$t = o \cdot \vec{n} \tag{49}$$

The solution for the center, c, is the following:

$$c = \begin{cases} \frac{(0, s_z z_y + s_y(1 - z_z), s_y y_z + s_z(1 - y_y))}{1 + x_x - y_y - z_z} & \text{if } 1 + x_x - y_y - z_z \neq 0 \\ \frac{(s_z z_x + s_x(1 - z_z), 0, s_x x_z + s_z(1 - x_x))}{1 - x_x + y_y - z_z} & \text{if } 1 - x_x + y_y - z_z \neq 0 \\ \frac{(s_y y_x + s_x(1 - y_y), s_x x_y + s_y(1 - x_x), 0)}{1 - x_x - y_y + z_z} & \text{if } 1 - x_x - y_y + z_z \neq 0 \end{cases} \tag{50}$$

$$\text{where } s = o - t\vec{n} \tag{51}$$

For any vector $\vec{v}$ non aligned with vector $\vec{n}$, the angle of rotation, θ, are given by the following:

$$\begin{aligned} \cos\theta &= \vec{u} \cdot M\vec{u} \\ \sin\theta &= (\vec{n} \times \vec{u}) \cdot M\vec{u} \end{aligned} \tag{52}$$

$$\text{where } \vec{u} = \frac{\vec{v} - (\vec{v} \cdot \vec{n})\vec{n}}{\|\vec{v} - (\vec{v} \cdot \vec{n})\vec{n}\|} \tag{53}$$

There are many solutions for the angle θ, given by $\theta + 2k\pi, k \in \mathbb{Z}$. With the matrix form, the original value of θ is lost.

The hexanion that describes a straight path between between position matrix M_0 and position matrix M_1 is computed as above, with M defined as follows:

$$M = M_1 M_0^{-1} \tag{54}$$

7 Conclusion

Hexanions are useful for computing positions of a rigid object along its trajectory:

$$u\langle\vec{\omega},\vec{m}\rangle, \quad u\in[0,1] \tag{55}$$

Hexanions can be used to interpolate two positions, where $\langle\vec{\omega}_0,\vec{m}_0\rangle$ and $\langle\vec{\omega}_1,\vec{m}_1\rangle$ may be obtained using the matrix-to-hexanion conversion:

$$(1-u)\langle\vec{\omega}_0,\vec{m}_0\rangle + u\langle\vec{\omega}_1,\vec{m}_1\rangle, \quad u\in[0,1] \tag{56}$$

Hexanions are very useful for weighted sums of rigid transformations [ACWK04]:

$$\frac{\sum_i \phi_i\langle\vec{\omega}_i,\vec{m}_i\rangle}{\sum_i \phi_i} \tag{57}$$

On a computing device, the most expensive part of hexanions is the computation of the exponential. Depending on the scenario, this computation can be done only once to obtain a 4×4 matrix, or does not need to be computed explicitly: to deform a point or a vector, a closed-form formula can be used.

Limitations: At the beginning, we argued that having the translation and rotation components merged in a single formalism what an advantage in terms of simplifying the handling of a twist. From the point of view of a user who wants precise control, this can be seen as an inconvenient: it prevents the choice of individual schemes to handle the interpolation of the translation and rotation part of the motion. It is however possible to express this separation in the hexanion formalism. The following describes a motion straight from a point, c, to another point, $p+\vec{m}$, accompanied with a rotation about a center of rotation c:

$$\langle\vec{\omega},(\langle 0,\vec{m}\rangle * c)\times\vec{\omega}\rangle * \langle 0,\vec{m}\rangle \tag{58}$$

Given the transformation, the description of the motion is straightforward:

$$\langle u\vec{\omega},(\langle 0,u\vec{m}\rangle * c)\times(u\vec{\omega})\rangle * \langle 0,u\vec{m}\rangle, \quad u\in[0,1] \tag{59}$$

From this and the above formulas we can develop a closed-form the trajectory, $C(p)$, of a point, p:

$$\begin{array}{rcl} C(p,u) & = & p+u\vec{m}+\frac{1-\cos\|u\vec{\omega}\|}{\|\vec{\omega}\|^2}\vec{\omega}\times\vec{\xi}+\frac{\sin\|u\vec{\omega}\|}{\|\vec{\omega}\|}\vec{\xi} \\ \text{where } \vec{\xi} & = & \vec{\omega}\times(p-c) \end{array} \tag{60}$$

A Exponential closed-form

Let us denote the matrix form of a hexanion $M=(\vec{\omega},\vec{m})$. The following is its exponential:

$$\begin{array}{rcl} \exp M & = & \sum_{k=0}^{\infty}\frac{M^k}{k!} \\ & = & I+M+\sum_{k=1}^{\infty}\frac{M^{2k}}{(2k)!}+\sum_{k=1}^{\infty}\frac{M^{2k+1}}{(2k+1)!} \end{array} \tag{61}$$

If M is a translation, then $\exp M = I+M$. Let us assume in the following that M is not a translation. Using the relation between a hexanion and its rotation component, the following equalities can be obtained:

$$\begin{array}{rcl} M^{2k+1} & = & (-\|\omega\|^2)^k Rot M \\ M^{2k} & = & (-\|\omega\|^2)^{k-1}M^2 \end{array} \tag{62}$$

In the exponential, the above can then be substituted for their values:

$$\exp M = I + M + M^2 \sum_{k=1}^{\infty} \frac{(-\|\omega\|^2)^{k-1}}{(2k)!} + RotM \sum_{k=1}^{\infty} \frac{(-\|\omega\|^2)^k}{(2k+1)!} \tag{63}$$

Since we assume that M is not a translation, $\vec{\omega}$ is different from zero, and terms can be factored out from the sums:

$$\exp M = I + M - \frac{1}{\|\omega\|^2} M^2 \sum_{k=1}^{\infty} \frac{(-\|\omega\|^2)^k}{(2k)!} + \frac{1}{\|\omega\|} RotM \sum_{k=1}^{\infty} \frac{\|\omega\|(-\|\omega\|^2)^k}{(2k+1)!} \tag{64}$$

In the first and second sums, parts of the cosine and sine of $\|\vec{\omega}\|$ can be identified:

$$\exp M = I + M - \frac{1}{\|\vec{\omega}\|^2} M^2 (\cos\|\vec{\omega}\| - 1) + \frac{1}{\|\omega\|} RotM (\sin\|\omega\| - \|\omega\|) \tag{65}$$

Using the relation between $RotM$ and M to replace $RotM$, we obtain the final formula:

$$\exp M = I + M + \frac{1 - \cos\|\vec{\omega}\|}{\|\vec{\omega}\|^2} M^2 + \frac{\|\omega\| - \sin\|\omega\|}{\|\omega\|^3} M^3 \tag{66}$$

In the case where M is a rotation, the formula further simplifies using the equality $M^3 = -\|\omega\|^2 M$.

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