The Structure of (some!) Permutation Classes

Michael Albert
Department of Computer Science
University of Otago
Dunedin, New Zealand
malbert@cs.otago.ac.nz

Paris, 12/11/08
Permutation and Subpermutation
Permutation and Subpermutation

5 3 7 2 6 8 1 9 4
Permutation and Subpermutation

\[
\begin{array}{ccccccc}
5 & 3 & 7 & 2 & 6 & 8 & 1 & 9 & 4
\end{array}
\]
Permutation and Subpermutation
Permutation and Subpermutation

1324 \preceq 537268194

1 3 2 4

1324 \preceq 537268194
Definition

If we construe permutations as sequences, then the involvement order on permutations is defined by:

\[ \alpha \preceq \beta \quad \text{iff} \quad \{ \beta \text{ contains a subsequence whose terms are in the same relative order as those of } \alpha \} \]

A permutation class, \( C \), is a down-closed set for \( \preceq \). Its basis \( X \) consists of the \( \preceq \)-minimal permutations not in \( C \), and then:

\[ C = \text{Av}(X) \overset{\text{def}}{=} \{ \beta : \forall \alpha \in X \alpha \not\preceq \beta \}. \]
The Questions

- Where do permutation classes come from?
The Questions

- Where do permutation classes come from?
- How can we describe them? (other than by giving a basis.)
The Questions

- Where do permutation classes come from?
- How can we describe them? (other than by giving a basis.)
- How many permutations of length $n$ does the class $Av(X)$ contain?
The Questions

- Where do permutation classes come from?
- How can we describe them? (other than by giving a basis.)
- How many permutations of length $n$ does the class $Av(X)$ contain?
- What algorithms can we apply to recognize elements in such a class? To construct them? To find maximal subpermutations of arbitrary ones that belong to a given class?
The Questions

- Where do permutation classes come from?
- How can we describe them? (other than by giving a basis.)
- How many permutations of length $n$ does the class $Av(X)$ contain?
- What algorithms can we apply to recognize elements in such a class? To construct them? To find maximal subpermutations of arbitrary ones that belong to a given class?
- ...
The Answers

- Far too few!
Far too few!

I will concentrate on trying to understand some sufficient conditions for when a class $C$ has sufficiently nice structure to answer some or all of the questions above.
The Answers

- Far too few!
- I will concentrate on trying to understand some sufficient conditions for when a class $C$ has sufficiently nice structure to answer some or all of the questions above.
- The idea is, where possible, to look for general results rather than specific $ad$ $hoc$ examples.
Far too few!

I will concentrate on trying to understand some sufficient conditions for when a class $C$ has sufficiently nice structure to answer some or all of the questions above.

The idea is, where possible, to look for general results rather than specific ad hoc examples.

But, we are not unhappy with solving specific examples if they are interesting!
So What is Structure?

- I was hoping you wouldn’t ask!
So What is Structure?

- I was hoping you wouldn’t ask!
- We know it when we see it – but typically identified with some clear understanding of what it means for a permutation to belong to a class $C$, which goes beyond saying “none of the patterns in $X$ occur.”
So What is Structure?

- I was hoping you wouldn’t ask!
- We know it when we see it – but typically identified with some clear understanding of what it means for a permutation to belong to a class $C$, which goes beyond saying “none of the patterns in $X$ occur.”

- Take inspiration from algebra, graph theory, model theory – search for constructions, building blocks, and relationships.
So What is Structure?

- I was hoping you wouldn’t ask!
- We know it when we see it – but typically identified with some clear understanding of what it means for a permutation to belong to a class $C$, which goes beyond saying “none of the patterns in $X$ occur.”
- Take inspiration from algebra, graph theory, model theory – search for constructions, building blocks, and relationships.
- It is probably not the case that there is a single correct notion of “structure” for permutation classes.
So What is Structure?

- I was hoping you wouldn’t ask!
- We know it when we see it – but typically identified with some clear understanding of what it means for a permutation to belong to a class $\mathcal{C}$, which goes beyond saying “none of the patterns in $X$ occur.”
- Take inspiration from algebra, graph theory, model theory – search for constructions, building blocks, and relationships.
- It is probably not the case that there is a single correct notion of “structure” for permutation classes.
- Time for some examples!
Av(312) = \{ \alpha \ 1 \ \beta : \alpha < \beta \text{ both avoiding } 312 \}
But also $\text{Av}(312)$ is . . .

- The set of permutations that can be produced from the input $123 \cdots n$ by a single pass through a stack (last in - first out.)
But also $Av(312)$ is . . .

- The set of permutations that can be produced from the input $123 \cdots n$ by a single pass through a stack (last in - first out.)
- In obvious one to one correspondence with Dyck words.
But also $Av(312)$ is . . .

- The set of permutations that can be produced from the input $123 \cdots n$ by a single pass through a stack (last in - first out.)
- In obvious one to one correspondence with Dyck words.
- Or with plane binary trees with $n$ nodes.
But also $\text{Av}(312)$ is . . .

- The set of permutations that can be produced from the input $123 \cdots n$ by a single pass through a stack (last in - first out.)
- In obvious one to one correspondence with Dyck words.
- Or with plane binary trees with $n$ nodes.
- Enumerated by the Catalan numbers.
$\text{Av}(321) = \{ \pi : \pi \text{ a union of two monotone sequences} \}$
But also $Av(321)$ is . . .

- The set of permutations that can be produced from the input $123 \cdots n$ by a single pass through a pair of queues in parallel, or a single queue plus a shortcut.
But also $Av(321)$ is . . .

- The set of permutations that can be produced from the input $123 \cdots n$ by a single pass through a pair of queues in parallel, or a single queue plus a shortcut.
-Enumerated by the Catalan numbers.
\[ \text{Av}(4123, 4132, 4213, 4231, 4312, 4321) \]

\[ \pi \in \text{Av}(4\bullet\bullet) \] if each symbol of \( \pi \) is among the three smallest of its suffix.
$\pi \in \text{Av}(4\cdots)$ if each symbol of $\pi$ is among the three smallest of its suffix.
But also $Av(4 \bullet \bullet \bullet)$ is . . .

- The set of permutations that can be produced from the input $123 \cdots n$ by a buffer capable of holding no more than three items at a time.
But also $\text{Av}(4 \bullet \bullet)$ is . . .

- The set of permutations that can be produced from the input $123 \cdots n$ by a buffer capable of holding no more than three items at a time.
- Enumerated by $3^{n-2} \times 2 \times 1$. 
But also $\text{Av}(4 \bullet \bullet)$ is . . .

- The set of permutations that can be produced from the input $123 \cdots n$ by a buffer capable of holding no more than three items at a time.
- Enumerated by $3^{n-2} \times 2 \times 1$.
- Easily encoded over a three symbol alphabet.
$\mathcal{C}$ is Finite

If, and only if, for some $n$ and $k$:

\[
123 \cdots n \notin \mathcal{C} \quad \text{and} \quad k(k-1) \cdots 321 \notin \mathcal{C}.
\]

(or, more briefly, for some $n$, neither $123 \cdots n$ nor $n \cdots 321$ is in $\mathcal{C}$.)

(Erdős-Szekeres).
Structure So Far

- Recursive structure – permutations in the class constructed from smaller permutations in the class ($Av(312)$ – generalization to come.)
Structure So Far

- Recursive structure – permutations in the class constructed from smaller permutations in the class ($\text{Av}(312)$ – generalization to come.)
- Building from simpler classes ($\text{Av}(321)$ consists of all merges of two increasing sequences.)
Structure So Far

- Recursive structure – permutations in the class constructed from smaller permutations in the class ($\mathcal{A}_v(312)$ – generalization to come.)

- Building from simpler classes ($\mathcal{A}_v(321)$ consists of all merges of two increasing sequences.)

- Permutations produced by “machines” (or data structures). So long as removing items always makes it easier to work the machine, these will always give classes. If operation sequences are uniquely (or nearly so) determined by their output that’s great. If not . . . (perhaps later!)

- Correspondences with languages over finite alphabets (via some sort of encoding.)

- Conditions on the basis to give structure.
Structure So Far

- Recursive structure – permutations in the class constructed from smaller permutations in the class ($\text{Av}(312)$ – generalization to come.)
- Building from simpler classes ($\text{Av}(321)$ consists of all merges of two increasing sequences.)
- Permutations produced by “machines” (or data structures). So long as removing items always makes it easier to work the machine, these will always give classes. If operation sequences are uniquely (or nearly so) determined by their output that’s great. If not . . . (perhaps later!)
- Correspondences with languages over finite alphabets (via some sort of encoding.)
Structure So Far

- Recursive structure – permutations in the class constructed from smaller permutations in the class ($\mathcal{Av}(312)$ – generalization to come.)
- Building from simpler classes ($\mathcal{Av}(321)$ consists of all merges of two increasing sequences.)
- Permutations produced by “machines” (or data structures). So long as removing items always makes it easier to work the machine, these will always give classes. If operation sequences are uniquely (or nearly so) determined by their output that’s great. If not . . . (perhaps later!)
- Correspondences with languages over finite alphabets (via some sort of encoding.)
- Conditions on the basis to give structure.
Block Decomposition
Block Decomposition

\[ 463592178 = 2413[2413, 1, 21, 12] \]
A permutation (of length \( n > 1 \)) is *simple* if there is no non-trivial proper interval whose image is also an interval.
Simple Permutations

- A permutation (of length \( n > 1 \)) is *simple* if there is no non-trivial proper interval whose image is also an interval.
- The first few: 12, 21, 2413, 3142, 24153, ...
A permutation (of length $n > 1$) is *simple* if there is no non-trivial proper interval whose image is also an interval.

The first few: 12, 21, 2413, 3142, 24153, ... 

Every permutation is the *inflation* of a unique simple permutation, called its *skeleton*. This is called its *block decomposition*. The blocks are also uniquely determined if the skeleton is not 12 or 21. In that case we can enforce uniqueness by requiring that the first block not be so decomposable (so 21354 = 12[21, 132].)
The total number of simple permutations of length $n$ is asymptotically $n!/e^2$, i.e. a positive proportion of all permutations are simple. But, it appears that in any infinite class $C$, the simple elements of $C$ have density 0.

Is that true? If so, why?
A class, $C$, is *wreath closed* if, whenever $\sigma$ (of length $k$) and $\pi_1, \pi_2, \ldots, \pi_k$ (of arbitrary lengths) are in $C$, then so is $\sigma[\pi_1, \pi_2, \ldots, \pi_k]$. 

Equivalently, every basis element of $C$ is simple.

Equivalently, $C$ is the closure of a downward closed set of simple permutations under inflation.
A class, \( C \), is **wreath closed** if, whenever \( \sigma \) (of length \( k \)) and \( \pi_1, \pi_2, \ldots, \pi_k \) (of arbitrary lengths) are in \( C \), then so is \( \sigma[\pi_1, \pi_2, \ldots, \pi_k] \).

Equivalently, every basis element of \( C \) is simple.
A class, $\mathcal{C}$, is *wreath closed* if, whenever $\sigma$ (of length $k$) and $\pi_1, \pi_2, \ldots, \pi_k$ (of arbitrary lengths) are in $\mathcal{C}$, then so is $\sigma[\pi_1, \pi_2, \ldots, \pi_k]$.

Equivalently, every basis element of $\mathcal{C}$ is simple.

Equivalently, $\mathcal{C}$ is the closure of a downward closed set of simple permutations under inflation.
If $C$ is a class with finitely many simple permutations then:

- $C$ has a finite basis,
- $C$ has an algebraic generating function
- and much much more (Brignall, Huczynska, Vatter)
Finitely Many Simples

If $C$ is a class with finitely many simple permutations then:

- $C$ has a finite basis,
If $\mathcal{C}$ is a class with finitely many simple permutations then:

- $\mathcal{C}$ has a finite basis,
- $\mathcal{C}$ has an algebraic generating function
If $\mathcal{C}$ is a class with finitely many simple permutations then:

- $\mathcal{C}$ has a finite basis,
- $\mathcal{C}$ has an algebraic generating function
- and much much more (Brignall, Huczynska, Vatter)
Furthermore

There is an effective procedure, given a finite basis $X$ to determine whether or not $Av(X)$ contains only finitely many simple permutations (Brignall, Ruškuc, Vatter).

This is based on the existence of certain unavoidable structures in large simple permutations, not unlike the Erdős-Szekeres characterization of finite classes.

Are classes with finitely many simples then “finis”?
Separable Permutations

- $S = Av(2413, 3142)$ is the wreath closure of 12 and 21. It is called the class of *separable* permutations, and is enumerated by the large Schroeder numbers.
Separable Permutations

- $S = \text{Av}(2413, 3142)$ is the wreath closure of 12 and 21. It is called the class of *separable* permutations, and is enumerated by the large Schroeder numbers.
- The degree over $\mathbb{Q}(t)$ of the generating function of any subclass of $S$ is a power of 2.
Separable Permutations

- $S = \text{Av}(2413, 3142)$ is the wreath closure of 12 and 21. It is called the class of *separable* permutations, and is enumerated by the large Schroeder numbers.
- The degree over $\mathbb{Q}(t)$ of the generating function of any subclass of $S$ is a power of 2.
- If $\pi_1 = 132$, and
  \[
  \pi_{n+1} = \begin{cases} 
  12[1, \pi_n] & \text{n even,} \\
  21[1, \pi_n] & \text{n odd.}
  \end{cases}
  \]
  (so: 132, 4132, 15243, 615243, ...) then the degree of the generating function of $\text{Av}(2413, 3142, \pi_n)$ over $\mathbb{Q}(t)$ is precisely $2^n$. 

But ...  

- Can we characterize exactly the subclasses of $S$ which have rational generating functions? (I think we’re close ... )
But . . .

- Can we characterize exactly the subclasses of $S$ which have rational generating functions? (I think we’re close . . .)
- What is the recipe that takes an input $X$ and gives us the degree of the generating function of $Av(X)$?
But . . .

- Can we characterize exactly the subclasses of $S$ which have rational generating functions? (I think we’re close . . . )
- What is the recipe that takes an input $X$ and gives us the degree of the generating function of $Av(X)$?
- What sorts of restrictions are there on these generating functions?
Can we characterize exactly the subclasses of $S$ which have rational generating functions? (I think we’re close . . . )

What is the recipe that takes an input $X$ and gives us the degree of the generating function of $A_v(X)$?

What sorts of restrictions are there on these generating functions?

What can be said about the set of growth rates of separable classes?
Another place to search for structure is to look for encodings of classes over finite alphabets.
Another place to search for structure is to look for encodings of classes over finite alphabets.

The \textit{k-rank bounded} permutations $\mathcal{B}_k = \text{Av}((k + 1) \bullet \cdots \bullet)$ are an obvious example.
Another place to search for structure is to look for encodings of classes over finite alphabets.

The \textit{k-rank bounded} permutations $B_k = \text{Av}((k + 1) \bullet \cdots \bullet)$ are an obvious example.

Alternatively, we can use “histoires de Laguerre” (X. Viennot), restricted in various ways.
Another place to search for structure is to look for encodings of classes over finite alphabets.

The *k-rank bounded* permutations $B_k = \text{Av}((k + 1) \bullet \cdots \bullet)$ are an obvious example.

Alternatively, we can use “histoires de Laguerre” (X. Viennot), restricted in various ways.

Other examples include the $W$-classes (where the number of monotone runs is bounded.)
A Metatheorem

Suppose that an encoding of a class $\mathcal{C}$ over a finite alphabet $\Sigma$ is such that the relation:

$$\sigma \preceq \pi$$

(for $\sigma, \pi \in \mathcal{C}$) is accepted by a finite state transducer.

Then a subclass of $\mathcal{C}$ is a regular set in $\Sigma^*$ if and only if its basis (relative to $\mathcal{C}$) is regular.

In particular, . . .
The required transducer commits in advance to which of the $k$ smallest remaining symbols must be deleted.

Think of its states as encoded by bit strings $b_1 b_2 \ldots b_k$ with $b_j = 1$ meaning “I promise to delete the $j$-th smallest remaining symbol.”

To process an input symbol, check first if it is to be deleted. If so, output nothing; if not, output its value minus the number of smaller items to be deleted. Then, in either case, eliminate its bit from the string, and add a new final bit of your choice.

Do some minor tinkering to handle end cases.
But . . .

- This does not mean that *all* rank-bounded classes have good structure.
But …

- This does not mean that all rank-bounded classes have good structure.
- Already, $\mathcal{B}_3$ contains an infinite antichain, and therefore has $2^{\aleph_0}$ subclasses not all of which can have good structure.

(A new question?) Is there a subclass of $\mathcal{B}_3$ with an algebraic, but not rational, generating function?
But . . .

- This does not mean that \textit{all} rank-bounded classes have good structure.
- Already, $\mathcal{B}_3$ contains an infinite antichain, and therefore has $2^{\aleph_0}$ subclasses not all of which can have good structure.
- But, every finitely based subclass of $\mathcal{B}_3$ has a rational generating function.

(A new question?) Is there a subclass of $\mathcal{B}_3$ with an \textit{algebraic}, but not rational, generating function?
But . . .

- This does not mean that *all* rank-bounded classes have good structure.
- Already, $\mathcal{B}_3$ contains an infinite antichain, and therefore has $2^{\aleph_0}$ subclasses not all of which can have good structure.
- But, every finitely based subclass of $\mathcal{B}_3$ has a rational generating function.
- (A new question?) Is there a subclass of $\mathcal{B}_3$ with an algebraic, but not rational, generating function?
Other sorts of structure

- A class, $\mathcal{C}$, is *atomic* (a.k.a. “has the joint embedding property”) if, for all $\alpha, \beta \in \mathcal{C}$, there exists $\pi \in \mathcal{C}$ with $\alpha, \beta \preceq \pi$. 

- Equivalently, there are two linear orders $D$ and $R$ and a bijection $\pi : D \rightarrow R$ such that $\mathcal{C}$ is the class of all patterns occurring in $\pi$. Call such a $\pi$ a *representation* of $\mathcal{C}$.

- We can ask: what extra properties can be demanded of a representation? For example, every class $\mathcal{C}$ has a representation with the property that any atomic subclass of $\mathcal{C}$ occurs as a “subrepresentation”. This is a simple application of the compactness theorem – but in case $\mathcal{C}$ has uncountably many such subclasses, can we always find a countable representation of this type?

- This is the model theoretic approach and is very much in its infancy.
Some other sorts of structure

- A class, $C$, is **atomic** (a.k.a. “has the joint embedding property”) if, for all $\alpha, \beta \in C$, there exists $\pi \in C$ with $\alpha, \beta \preceq \pi$.

- Equivalently, there are two linear orders $D$ and $R$ and a bijection $\pi : D \to R$ such that $C$ is the class of all patterns occurring in $\pi$. Call such a $\pi$ a **representation** of $C$. 

We can ask: what extra properties can be demanded of a representation? For example, every class $C$ has a representation with the property that any atomic subclass of $C$ occurs as a “subrepresentation”. This is a simple application of the compactness theorem – but in case $C$ has uncountably many such subclasses, can we always find a countable representation of this type? 

This is the model theoretic approach and is very much in its infancy.
Other sorts of structure

A class, $C$, is *atomic* (a.k.a. “has the joint embedding property”) if, for all $\alpha, \beta \in C$, there exists $\pi \in C$ with $\alpha, \beta \preceq \pi$.

Equivalently, there are two linear orders $D$ and $R$ and a bijection $\pi : D \rightarrow R$ such that $C$ is the class of all patterns occurring in $\pi$. Call such a $\pi$ a *representation* of $C$.

We can ask: what extra properties can be demanded of a representation? For example, every class $C$ has a representation with the property that any atomic subclass of $C$ occurs as a “subrepresentation”. This is a simple application of the compactness theorem – but in case $C$ has uncountably many such subclasses, can we always find a countable representation of this type?
Other sorts of structure

- A class, $\mathcal{C}$, is **atomic** (a.k.a. “has the joint embedding property”) if, for all $\alpha, \beta \in \mathcal{C}$, there exists $\pi \in \mathcal{C}$ with $\alpha, \beta \preceq \pi$.

- Equivalently, there are two linear orders $D$ and $R$ and a bijection $\pi : D \rightarrow R$ such that $\mathcal{C}$ is the class of all patterns occurring in $\pi$. Call such a $\pi$ a **representation** of $\mathcal{C}$.

- We can ask: what extra properties can be demanded of a representation? For example, every class $\mathcal{C}$ has a representation with the property that any atomic subclass of $\mathcal{C}$ occurs as a “subrepresentation”. This is a simple application of the compactness theorem – but in case $\mathcal{C}$ has uncountably many such subclasses, can we always find a countable representation of this type?

- This is the model theoretic approach and is very much in its infancy.
Are $\text{Av}(312)$ and $\text{Av}(321)$ different?

Well . . .
Are $\text{Av}(312)$ and $\text{Av}(321)$ different?

Well . . .

Yes!
Are $\text{Av}(312)$ and $\text{Av}(321)$ different?

Well . . .

Yes!

- The first is a subclass of $S$ and hence contains no infinite antichains, while the second does.
Are $\text{Av}(312)$ and $\text{Av}(321)$ different?

Well . . .

Yes!

- The first is a subclass of $S$ and hence contains no infinite antichains, while the second does.
- Every proper subclass of the first has a rational generating function.
Are $\text{Av}(312)$ and $\text{Av}(321)$ different?

Well . . .

Yes!

- The first is a subclass of $S$ and hence contains no infinite antichains, while the second does.
- Every proper subclass of the first has a rational generating function.
- Not much (really, next to nothing) is known about the behaviour of even finitely based subclasses of the second.
The Frontier

Is wide open:

- What about $\text{Av}(4231)$?
- What about “simple” machines (two stacks in series, for example.)
- What about detailed understanding of $S$? (prediction of degree of algebraicity without computation; characterization of growth rates, . . . )
- How well can we “approximate” arbitrary classes with ones having structure?