Double bubble trouble

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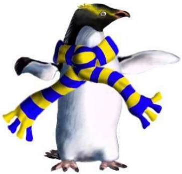

http://arxiv.org/abs/1008.5299

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SAPERE AUDE

Te Whare Wānanga o Otāgo

NEW ZEALAND

Stack sorting

- A *stack* is a last in/first out data structure
- If we present data to a stack, popping if the newly arrived data is larger than the stack top, and pushing otherwise then we sort some input sequences
- Precisely, we sort Av(231)
- If we think of the input as a permutation of length n, written in one line notation as $\alpha n \beta$ then the effect of a pass through a stack operating above is:

$$S(\alpha n \beta) = S(\alpha) S(\beta) n$$

Bubble sort

- *Bubble sort* is a familiar elementary sorting algorithm
- In a single pass, we always 'hold' the current maximum element
- When a smaller element is encountered it is simply output, when a larger element is encountered it replaces the current maximum (which is output)
- Essentially, we have a buffer capable of holding two items and an atomic step consists of outputting the smaller of the two (if the buffer is full), and replacing it by the next input item
- Functionally:

$$B(\alpha n \beta) = B(\alpha)\beta n$$

Bubble sort sorts?

Recall:

$$B(\alpha n \beta) = B(\alpha)\beta n$$

- For the RHS to be sorted, β must be increasing and consist of an interval of values from $|\alpha| + 1$ through $n - 1$
- And of course α must be sortable
- So
$$B^{-1}(\mathrm{Av}(21)) = \bigoplus \{1, 21, 312, 4123, \ldots\}$$
- Alternative:
$$B^{-1}(\mathrm{Av}(21)) = \mathrm{Av}(231, 321)$$
- This contains 2^{n-1} permutations of length n (one to one correspondence with compositions)
- So, in some sense, B is optimal in as much as it performs only $n - 1$ comparisons

Double bubble sorts?

- The sequence sorted by $S \circ S$ are not a classical pattern class (they can be characterized in terms of barred pattern avoidance – West)
- For B the situation is more clear cut, which is easily seen by chaining together two element buffers
- A chain of k two element buffers behaves like a $k + 1$ element buffer
- So long as the smallest remaining item is always among the first $k + 1$ elements B^k will sort correctly, i.e.

$$B^{-k}(\mathrm{Av}(21)) = \mathrm{Av}(\mathcal{S}_{k+1} \ominus 1)$$

- Unfortunately, optimality is gone – this class has growth rate $k + 1$ while requiring $O(kn)$ comparisons

More generally

- As noted above, $S^{-2}(\text{Av}(21))$ is not a classical pattern class, or put another way $S^{-1}(\text{Av}(231))$ is not a pattern class
- Whenever we have a *sorting operator* (whatever that means), we might consider the general problem of its (inverse) effect on pattern classes
- For example, what permutations are sorted by an application of B and then one of S?

$$(S \circ B)^{-1}(\text{Av}(21)) = B^{-1} \circ S^{-1}(\text{Av}(21)) = B^{-1}(\text{Av}(231))$$

All about B

Theorem

Let π be a single permutation. Then, $B^{-1}(\mathrm{Av}(\pi))$ is a pattern class if and only if:

- *π has two or fewer left to right maxima, or*
- *π has exactly three left to right maxima the last of which is the final element of π.*

Proof.

In 20+ minutes? Don't be silly. $\square$

Remarks on the theorem

- In the positive cases it is actually possible to explicitly compute bases
- It is not *automatically* possible to compute enumerations or growth rates
- For $B^{-1}(\text{Av}(231))$ the basis is $\{3241, 2341, 4231, 2431\}$
- The number of permutations sorted by "*B* then *S*" is actually $\binom{2n-2}{n-1}$, so we don't actually gain on *S* alone in growth rate

Points to ponder

- Just what is a *sorting operator*? What is the weakest useful definition? What are the strongest reasonable properties? Straw men:
  - A length preserving map $X : \mathcal{S} \to \mathcal{S}$ is a *sorting operator* if the inverted values of $X(\pi)$ are always a subset of those of π
  - A sorting operator X is *blocking* if for every σ such that some inversion of σ is not repaired by X, then for every π containing σ the corresponding inversion of π is not repaired by X
- What are the nice, but non-trivial, examples of operators X such that $X^{-1}(\mathcal{C})$ is always a pattern class when $\mathcal{C}$ is?
- Are there nice asymptotically optimal operators more powerful than B?