Asymptotic and exact enumeration of permutation classes

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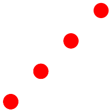
Example 21

Question

How many permutations of length n contain no decreasing subsequence of length two?

Consider describing the plot of such a permutation from left to right.



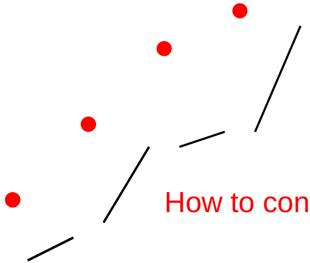
Too easy!

Example 321

Question

How many permutations of length n contain no decreasing subsequence of length three?

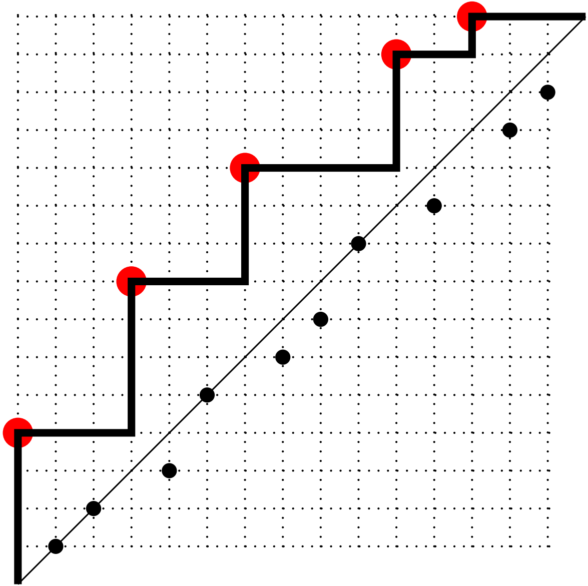
Consider describing the plot of such a permutation from left to right.



How to convert to an enumeration?

The first element is followed by some smaller stuff in increasing order, a new maximum, more smaller stuff (still increasing) and so on . . .

Example 321 continued

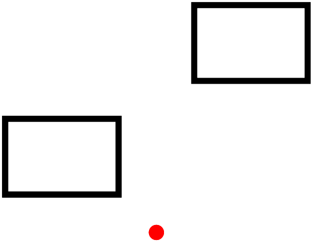


Example 312

Question

How many permutations of length n contain no subsequence cab where a $<$ b $<$ c?

Consider describing the plot of such a permutation from bottom to top.



The least element is flanked to its left and right by sequences of the same type, with all the elements on the left being less than those on the right, so $f = 1 + fxf$

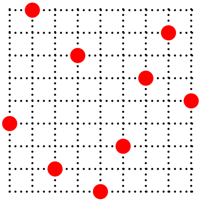
Observations

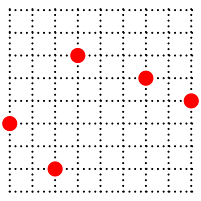
- Both examples 321 and 312 produced the same enumeration sequence, namely the Catalan numbers
- The reason this was true for 321 was subtle
- Or, at least more subtle than the reason it was true for 312
- What about the other examples: 123, 132, 213, 231?
- All those are the same as one of the ones we did due to obvious symmetries (stand on your head, look in a mirror)

Permutation classes

Definition

A permutation class is a collection of permutations, $\mathcal{C}$, with the property that, if $\pi \in \mathcal{C}$ and we erase some points from its plot, then the permutation defined by the remaining points is also in $\mathcal{C}$.





$492713685 \in \mathcal{C}$ implies $21543 \in \mathcal{C}$

Permutation classes

- In other words, permutation classes are simply downward closed sets in the *subpermutation* ordering
- They can be defined in various ways including as in our examples "permutations that *do not* contain 321" (or 312 or whatever)
- That is, $\mathcal{C} = \mathrm{Av}(X)$ for those permutations that *avoid* all the patterns in X (if X is an antichain it is called the *basis* of $\mathcal{C}$)
- The name of the game is to try to *understand* the structure of permutation classes (or to identify when this is possible)
- Enumeration is a consequence or symptom of such understanding

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First steps

- It seems plausible that the fewer permutations there are in a class, or alternatively the more restrictive the conditions of membership, then the more likely it is to have good structure
- Simion and Schmidt (1985) successfully enumerated all classes Av(X) for $X \subseteq \mathcal{S}_3$
- For a single avoided pattern of length 4, it turns out there are only three possible enumerations (based on both specific and general symmetries)
- Av(1234) was enumerated by Gessel (1990)
- Av(1342) by Bóna (1997)
- Of Av(1324) we shall speak later
- For single element bases of length 5 or more, nothing much is known in the non-monotone case
- All doubleton bases of lengths 4 and 3, and many of length 4 and 4 are known

Stanley-Wilf conjecture

Relative to the set of all permutations, proper permutation classes are small. Specifically:

Theorem

Let $\mathcal{C}$ be a proper permutation class. Then, the growth rate of $\mathcal{C}$,

$$\mathrm{gr}(\mathcal{C}) = \limsup |\mathcal{C} \cap \mathcal{S}_n|^{1/n}$$

is finite.

This was known as the *Stanley-Wilf conjecture* and it was proven in 2004 by Marcus and Tardos.

Are growth rates limits?

- Arratia (1999) proved that in the case $\mathcal{C} = \text{Av}(\pi)$, gr$(\mathcal{C})$ is always a limit (not just a limit superior)
- His argument generalizes to many other classes

Conjecture

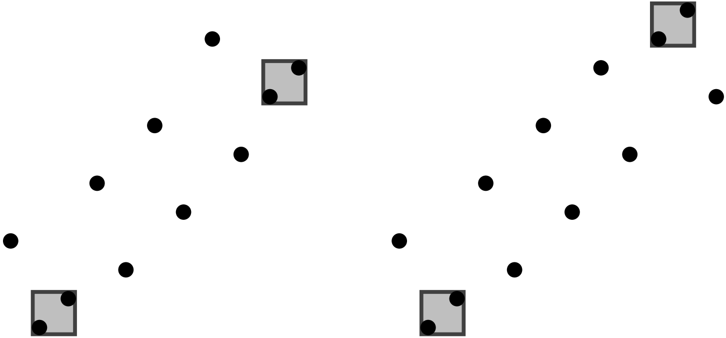
Every class has a limiting growth rate.

The obvious next questions are:

- What growth rates can occur?
- What can be said about classes of particular growth rates?

Antichains

The subpermutation order contains infinite antichains.



Consequently, there exist $2^{\aleph_0}$ distinct enumeration sequences for permutation classes – we must be careful not to try to do too much.

Small growth rates

- Kaiser and Klazar (2003) showed that the only possible values of $\mathrm{gr}(\mathcal{C})$ less than 2 are the greatest positive solutions of:

$$x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = 0$$

- Vatter (2010) showed that every real number larger than the unique real solution (approx. 2.482) of

$$x^5 - 2x^4 - 2x^2 - 2x - 1 = 0$$

  occurs as a growth rate
- V (PLMS, to appear) further showed that the smallest growth rate of a class containing an infinite antichain is the unique positive solution, $\kappa \simeq 2.20557$ of

$$x^3 - 2x^2 - 1 = 0$$

  and completely characterized the set of possible growth rates below κ

Where does that leave us?

- There's a bit of a grey area between 2.2 and 2.5
- These answers are 'what growth rates are possible?'
- What about 'given a class, determine its growth rate'
- Again because of the equation:

$$2^{\aleph_0} = \text{chaos}$$

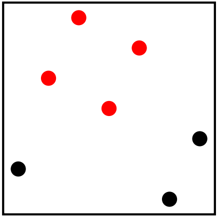
we will need to insist on some restrictive conditions

Simple permutations

Definition

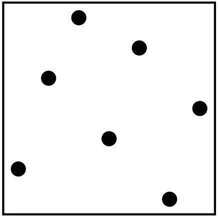
A permutation is simple if it contains no nontrivial consecutive subsequence whose values are also consecutive (though not necessarily in order)

2 5 7 4 6 1 3



Not simple

2 5 7 3 6 1 4



Simple

Simple permutations

Definition

A permutation is simple if it contains no nontrivial consecutive subsequence whose values are also consecutive (though not necessarily in order)

- Simple permutations form a positive proportion of all permutations (asymptotically $1/e^2$)
- In many (conjecturally all) proper permutation classes they have density 0
- We can hope to understand a class by understanding its simples and how they *inflate*
- Specifically, this may yield functional equations of the generating function and hence computations of the enumeration and/or growth rate

Finitely many simple permutations

Theorem

If a class has only finitely many simple permutations then it has an algebraic generating function.

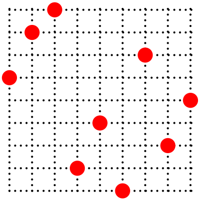
- A and Atkinson (2005)
- Effective ‘in principle’, i.e. an algorithm for computing a defining system of equations for the generating function
- Some interesting corollaries, e.g. if a class has finitely many simples and does not contain arbitrarily long decreasing permutations then it has a rational generating function
- “The prime reason for giving this example is to show that we are not necessarily stymied if the number of simple permutations is infinite.”

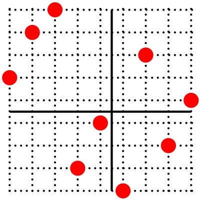
Encodings

- The enumerative combinatorics of words and trees is better understood
- So, apply leverage from there by encoding permutation classes
- For trees this gives the *generating tree* or Italian school *ECO* approach
- For words, A, At and Ruškuc (2003) give (fairly) general criteria for the encoding of a permutation class by a regular language
- Extended to a new type of encoding, the *insertion encoding* (prefigured by Viennot) by A, Linton and R (2005)

Grid classes

- The notion of *griddable* class was central to V's characterization of small permutation classes
- Loosely, a griddable class is associated with a matrix whose entries are (simpler) permutation classes
- All permutations in the class can be chopped apart into sections that correspond to the matrix entries





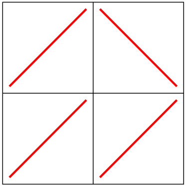
Geometric monotone grid classes

In a *geometric* grid class, the permutations need to be drawn from the points of a particular representation in $\mathbb{R}^2$

Theorem (A, At, Bouvel, R and V (submitted))

Every geometrically griddable class:

- *is partially well ordered;*
- *is finitely based;*
- *is in bijection with a regular language and thus has a rational generating function.*



Small classes

A *strongly rational* class is one all of whose subclasses have rational generating functions.

Theorem (A, R and V (11/11))

If $\mathcal{C}$ is a geometrically griddable class, and $\mathcal{U}$ is a strongly rational class, then the class of all inflations of permutations in $\mathcal{C}$ by permutations in $\mathcal{U}$ is also strongly rational.

Corollary

Every small class (of growth rate less than $2.20\ldots$) has a rational generating function.

The separable world

- *Separable* permutations are recursively produced from 1 by stacking graphs above/below and right/left of one another
- This is the largest class without any nontrivial simple permutations
- It contains Av(312) (and all of its symmetries)
- A, At and V (2011) showed that any subclass that does not contain Av(312) or one of its symmetries necessarily has a rational generating function
- Conjecture: the converse is also true
- Every subclass has a generating function which is algebraic of degree 2^d for some d
- Do these form some sort of hierarchy of complexity?

Av(4231)

- An annoying itch - the last remaining length four permutation such that the growth rate of the class avoiding it is unknown
- A et al. (2006) used an idea of “approximating” this class by regular subclasses to obtain lower bounds on the growth rate
- Using a resulting automaton with 2.4×10^7 states gave a lower bound of 9.47 which at least refuted a conjecture of Arratia
- Actual answer (based on trend guessing) probably between 11 and 12
- Recently (Nov 24, arXiv:1111.5736), Claesson, Jelínek and Steingrímsson improved the known upper bounds to 16, and gave a conjectural argument that would provide an upper bound of $e^{\pi\sqrt{2/3}} \sim 13.002$

Things we haven't talked about

Mostly due to lack of knowledge!

- Given a class, what does a *typical* permutation in the class look like? Can we generate random elements of the class quickly?
- Given $\mathcal{C}$, can we provide guarantees that for some suitably chosen classes $\mathcal{C}^{(i)}$,

$$\lim_{i \to \infty} \mathrm{gr}(\mathcal{C}^{(i)}) = \mathrm{gr}(\mathcal{C})?$$

- Are all growth rates really limits?
- Many variations on the "longest increasing subsequence" theme
- To what extent do we *really* understand the "fine structure" of (say) subclasses of the separable permutations?