# Separable permutations

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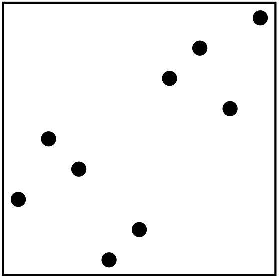
# A separable permutation

3 5 4 1 2 7 8 6 9



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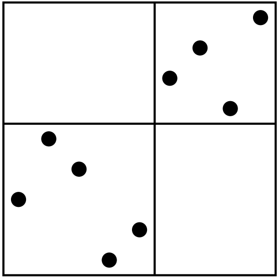

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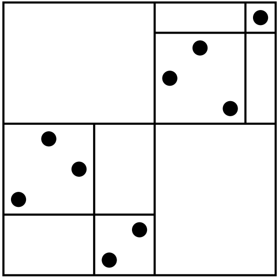

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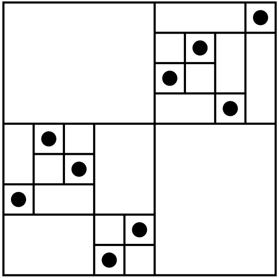
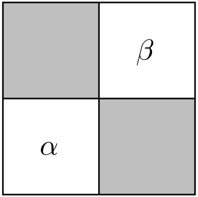
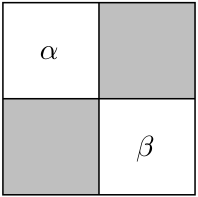


# Two operations on permutations

$\alpha \oplus \beta =$ 


$\alpha \ominus \beta =$ 


The *separable* permutations, $\mathcal{S}$, are the closure of $\{1\}$ under $\oplus$ and $\ominus$.

# Observations about $\mathcal{S}$

- If $\pi$ is separable, and you erase some points from its graph the resulting permutation is also separable ($\mathcal{S}$ is a *permutation class*).
- Every permutation of length $\leq 3$ is separable, only two of length four are not (2413 and 3142).
- A permutation is separable if and only if it contains no four element subsequence whose relative ordering matches 2413 or 3142 (i.e. it *avoids* these two permutations).
- The separable permutations are enumerated by the large Schröder numbers:

$$S(t) = \frac{1 - t - \sqrt{1 - 6t + t^2}}{2t}.$$

- Every subclass of $\mathcal{S}$ has an algebraic generating function of degree a power of 2 over $\mathbb{Q}(t)$.

# Special subclasses of $\mathcal{S}$

- The class $\mathcal{S}$ is the smallest solution of the equation:

$$\mathcal{S} = \mathcal{S} \oplus \mathcal{S} = \mathcal{S} \ominus \mathcal{S}.$$

- We get four different classes by changing one of the terms on the right hand side to 1 (i.e. $\{1\}$)

$$\begin{aligned} \mathcal{A} &= \mathcal{A} \oplus \mathcal{A} = \mathcal{A} \ominus 1, \\ \mathcal{B} &= \mathcal{B} \oplus \mathcal{B} = 1 \ominus \mathcal{B}, \\ \mathcal{C} &= \mathcal{C} \oplus 1 = \mathcal{C} \ominus \mathcal{C}, \\ \mathcal{D} &= 1 \oplus \mathcal{D} = \mathcal{D} \ominus \mathcal{D}. \end{aligned}$$

- These turn out to be the four classes defined by avoiding a single non-monotone permutation of length 3.
- Each is enumerated by the Catalan numbers.

# One more class

- The class $\mathcal{X}$ is the smallest class satisfying:

$$\mathcal{X} = 1 \oplus \mathcal{X} = \mathcal{X} \oplus 1 = 1 \ominus \mathcal{X} = \mathcal{X} \ominus 1.$$

- It has a *rational* generating function:

$$X(t) = \frac{x - 2x^2}{1 - 4x + 2x^2}.$$

- It is also defined as the set of permutations avoiding all of 2143, 2413, 3142, 3412.
- Any, and only, permutations in $\mathcal{X}$ can be drawn up to rescaling of axes on the lines $y = \pm x$.

# Some important concepts

- Permutations are ordered by *involvement*, where $\alpha \leq \beta$ if there is a subsequence of $\beta$ with the same relative ordering as $\alpha$.
- A class is *partially well ordered* if it contains no infinite antichain of permutations.
- A class is *atomic* if it has the joint embedding property (i.e. for any $\alpha$, $\beta$ in the class, there is a $\pi$ which involves both).
- A class is *strongly rational* if it, and all of its subclasses have rational generating functions.
- An *inflation* of a permutation is obtained by replacing each of its elements by permutations. The notation $\mathcal{A}[\mathcal{B}]$ represents the set of all inflations of elements of $\mathcal{A}$ by elements of $\mathcal{B}$.

# Results

**Theorem**

*If $\mathcal{U}$ is a strongly rational class then so is $\mathcal{X}[\mathcal{U}]$.*

**Theorem**

*If $\mathcal{T}$ is a subclass of $\mathcal{S}$ then either $\mathcal{T}$ is strongly rational, or it contains one of the four classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$.*

The proof is a minimal counterexample argument. If such a counterexample existed it would have to be atomic. In that case, we can prove that $\mathcal{T} = \mathcal{X}[\mathcal{U}]$ for some proper subclass $\mathcal{U}$ of $\mathcal{T}$, yielding a contradiction.