Separable permutations

M. Albert, M. Atkinson, V. Vatter

Department of Computer Science, University of Otago

BCC2011

BLMS 2011, doi: 10.1112/blms/bdr022
A separable permutation

\[3\ 5\ 4\ 1\ 2\ 7\ 8\ 6\ 9\]
A separable permutation

3 5 4 1 2 7 8 6 9
A separable permutation

3 5 4 1 2 7 8 6 9
A separable permutation

3 5 4 1 2 7 8 6 9
Two operations on permutations

\[ \alpha \oplus \beta = \begin{array}{cc} & \beta \\ \alpha & \end{array} \]

\[ \alpha \ominus \beta = \begin{array}{cc} \alpha & \\ & \beta \end{array} \]

The separable permutations, \( S \), are the closure of \( \{1\} \) under \( \oplus \) and \( \ominus \).
Observations about $\mathcal{S}$

- If $\pi$ is separable, and you erase some points from its graph the resulting permutation is also separable ($\mathcal{S}$ is a permutation class).
- Every permutation of length $\leq 3$ is separable, only two of length four are not (2413 and 3142).
- A permutation is separable if and only if it contains no four element subsequence whose relative ordering matches 2413 or 3142 (i.e. it avoids these two permutations).
- The separable permutations are enumerated by the large Schröder numbers:
  $$S(t) = \frac{1 - t - \sqrt{1 - 6t + t^2}}{2t}.$$  
  
- Every subclass of $\mathcal{S}$ has an algebraic generating function of degree a power of 2 over $\mathbb{Q}(t)$. 
Special subclasses of $S$

- The class $S$ is the smallest solution of the equation:

$$S = S \oplus S = S \ominus S.$$ 

- We get four different classes by changing one of the terms on the right hand side to 1 (i.e. $\{1\}$)

$$A = A \oplus A = A \ominus 1,$$
$$B = B \oplus B = 1 \ominus B,$$
$$C = C \oplus 1 = C \ominus C,$$
$$D = 1 \oplus D = D \ominus D.$$ 

- These turn out to be the four classes defined by avoiding a single non-monotone permutation of length 3.

- Each is enumerated by the Catalan numbers.
One more class

- The class $\mathcal{X}$ is the smallest class satisfying:

$$\mathcal{X} = 1 \oplus \mathcal{X} = \mathcal{X} \oplus 1 = 1 \ominus \mathcal{X} = \mathcal{X} \ominus 1.$$ 

- It has a rational generating function:

$$X(t) = \frac{x - 2x^2}{1 - 4x + 2x^2}.$$ 

- It is also defined as the set of permutations avoiding all of $2143, 2413, 3142, 3412$.  

- Any, and only, permutations in $\mathcal{X}$ can be drawn up to rescaling of axes on the lines $y = \pm x$. 

Some important concepts

- Permutations are ordered by *involvement*, where $\alpha \leq \beta$ if there is a subsequence of $\beta$ with the same relative ordering as $\alpha$.
- A class is *partially well ordered* if it contains no infinite antichain of permutations.
- A class is *atomic* if it has the joint embedding property (i.e. for any $\alpha, \beta$ in the class, there is a $\pi$ which involves both).
- A class is *strongly rational* if it, and all of its subclasses have rational generating functions.
- An *inflation* of a permutation is obtained by replacing each of its elements by permutations. The notation $\mathcal{A}[\mathcal{B}]$ represents the set of all inflations of elements of $\mathcal{A}$ by elements of $\mathcal{B}$.
Results

Theorem

If $U$ is a strongly rational class then so is $X[U]$.

Theorem

If $T$ is a subclass of $S$ then either $T$ is strongly rational, or it contains one of the four classes $A, B, C, D$.

The proof is a minimal counterexample argument. If such a counterexample existed it would have to be atomic. In that case, we can prove that $T = X[U]$ for some proper subclass $U$ of $T$, yielding a contradiction.