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Technical note

On calculating the finite centre of rotation for rigid planar motion

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Abstract

In this technical note, a simpler least squares derivation for calculating the angle of rotation and finite centre of rotation of a set of marker points undergoing rigid planar rotation and translation is shown. The major advantage of the approach, other than the simple derivation, is the automatic inclusion of the calculation of a scaling factor between the two point sets - the calculation of which was not obvious in previous approaches [Challis J. Estimation of the finite center of rotation in planar movements. Med Eng Phys 2001;23(3):227–33, Spoor C, Veldpaus F. Rigid body motion calculated from spatial coordinates of markers. J Biomech 1980;13:391–3]. The final numerical calculations are similar to those of [Challis J. Estimation of the finite center of rotation in planar movements. Med Eng Phys 2001;23(3):227–33] and are trivial to implement. A matlab routine for computing the two quantities, and the scaling factor, is included. We demonstrate the method on a clinical example using lateral radiographs of the lumbar spine.

Keywords: Finite centre of rotation

1. Introduction

The finite centre of rotation (FCR) is a common measurement used for clinical diagnosis of joint function [1] including spine problems [3,4]. The FCR is often used as an approximation to the centre of rotation of a joint, and can be defined as the point which is unchanged by a rigid transformation involving a translation and a rotation. Such a transformation can be completely described by the FCR and an angle of rotation. The method of [5] is probably the best known of the techniques and despite its suboptimal error handling performance ([6]), it is still often the method of choice due to its simplicity ([7]). Other more robust techniques have also been proposed including those of [1,6,8]. Most of these use various least-squares formalisms with the method of [1] being the method of choice for 2D in the plane rotations with an axis of rotation orthogonal to the image plane. If the axis of rotation is not known, more complicated procedures such as those of [2,9,10] are available. However, the derivation of such methods is quite complex and regardless, as pointed out by [1] there is still a need for simple methods with a known axis of rotation. It is the method of [1] which we focus on in this paper. It is essentially a 2D simplification of the method of [9]. In our case we do not assume an a priori known rigid body configuration as in [1,9], since in some applications it is impractical to obtain such knowledge, when, for example, marker points are estimated from bone features in radiographs ([11,12]). The methods of [1,9] are easily adapted to the situation where there is no rigid body coordinate system. In this note we offer a trivial least squares derivation of the angle of rotation that explicitly provides a measure for the amount of scaling involved in the transformation and a complete matlab implementation of the method. These methods are essentially simplifications of the method of [10] for 2D rotations.

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2. Method

2.1. Calculating the angle of rotation

In [1], the optimal rotation is calculated as a two-step process. First, the rotation from rigid-body reference frame to inertial frame is calculated in both sets of markers. Then the rotation from one set of points to the other is calculated in a second step. However, in many 2D applications, a rigid frame of reference is unnecessary and the problem can be reformulated using markers measured at two points in time, t_1 and t_2.

Consider a rigid body undergoing a rotation and a translation from time t_1 to time t_2. Then in a single inertial reference frame, the coordinates of marker points on the rigid body are related by the equation:

$$p_i(t_2) = R(\theta)p_i(t_1) + v, \tag{1}$$

where $p_i(t_j) \in P(t_j)$ are the coordinates of marker point i at time t_j, $R(\theta)$ is a 2×2 rotation matrix, given by:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \tag{2}$$

and v is the translation vector of the transformation.

Rather than the explicit least squares derivation of [1], here we show a simpler and more elegant derivation that makes use of complex numbers instead of 2D coordinates. First, we form two mean adjusted complex vectors:

$$F = \begin{bmatrix} c_1(t_1) - \mu(t_1) \\ c_2(t_1) - \mu(t_1) \\ \vdots \\ c_n t_1 - \mu(t_1) \end{bmatrix}, E = \begin{bmatrix} c_1(t_2) - \mu(t_2) \\ c_2(t_2) - \mu(t_2) \\ \vdots \\ c_n(t_2) - \mu(t_2) \end{bmatrix} \tag{3}$$

where $c_k(t_i) = x_k(t_i) + \imath y_k(t_i)$, $p_k(t_1) = (x_k(t_i), y_k(t_i))$, and $\mu(t_i) = \sum c_i(t_i)/n$ is the mean of the marker points at time t_i.

To find the rotation, θ, that will maximally align one matrix with the other we wish to find the least squares solution to the following equation:

$$F = e^{\imath\theta}E, \tag{4}$$

since a rotation in the complex plane is specified by a multiplication with a unit complex number. Alternatively, we can write:

$$F = aE, \tag{5}$$

where a is the (complex) parameter that we can estimate in a straightforward manner using least squares techniques:

$$a = E^*F \tag{6}$$

where E^* is the pseudo-inverse of E and can be calculated from the singular value decomposition (SVD). Since E and F are complex vectors, in general a will be a complex number, $a = x + \imath y$, and θ can be extracted simply as $\theta = \tan^{-1}(y/x)$. The translation vector v, can then be calculated using Eq. (1).

The actual calculations performed are similar to those of [1], however the derivation is conceptually simpler. Also we now have a natural measure of contraction/expansion of the two sets of marker points. Theoretically, a should be a unit complex number - that is $x^2 + y^2 = 1$, however, in practice, this will not be exactly true and thus provides a measure of departure from unit scaling. Such an observation is less clear with the derivation of [1] and needed to be explicitly included in the least squares formulation of [10].

The use of the pseudo-inverse produces a solution that minimises the Euclidean distances between the point set F and the set aE [13]. As such it is robust to small deformations in the point sets that might arise from measurement error or imaging artefacts. Large deformations are likely to produce scale factors very different to unity. Alternatively we can use the least-squares residual [13] as an indication of the departure from rigid motion:

$$\rho = \|(I - EE^*)F\| \tag{7}$$

Further, we can calculate the FCR using the knowledge that it is unchanged by the transformation in a similar fashion to [8]. That is:

$$f = Rf + v, \tag{8}$$

or

$$f = (I - R)^{-1}v, \tag{9}$$

where I is the 2×2 identity matrix. The right hand side of Eq. (9) can be solved symbolically ([8]), giving the following solution:

$$f = \begin{bmatrix} \frac{1}{2}\left(x_v - \frac{y_v}{\tan(\frac{\theta}{2})}\right) \\ \frac{1}{2}\left(y_v + \frac{x_v}{\tan(\frac{\theta}{2})}\right) \end{bmatrix} \tag{10}$$

Eq. (8) is the 2D equivalent to the 3D condition described by [2](Eq. (26)) and the derivation is significantly simpler. The final calculations are similar to those used by [9] and [1] which were based on the derivation of [2].

3. Clinical example

The development of this method has been motivated by a need for estimating the FCR of lumbar spinal motion segments of patients with lower back pain in order to characterise differences that may be associated with these patients compared with normal subjects. Two lateral radiographs of each patient in full extension and full flexion were used for calculating the FCR. We use the protocol established by [14,15] for locating four feature points per vertebra which can then be used for calculating the FCR. Although [14,15] did not calculate the FCR in their work, we have used their feature point detection protocol since this is the most rigorously studied in

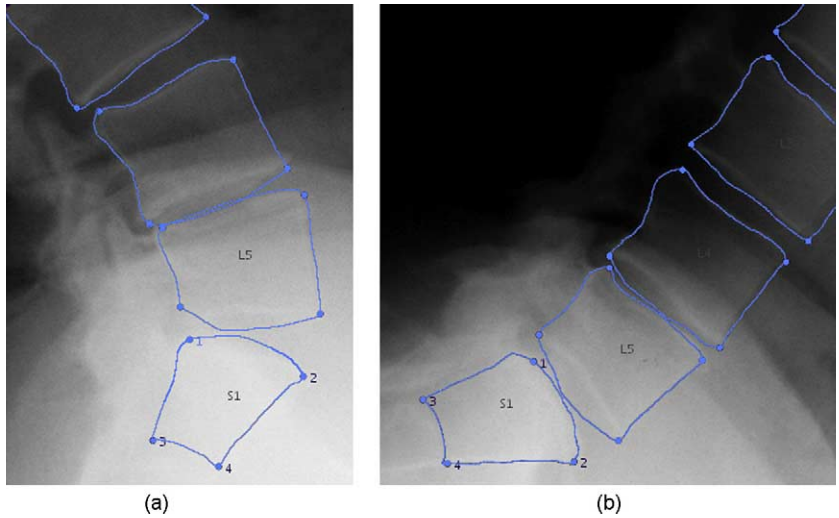

Fig. 1. An example radiographs and the associated feature points. (a) Extension image; (b) Flexion image.

terms of errors and is the current best practice method in the literature. Fig. 1 shows an example of the extracted feature points according to the method of [14,15].

The results for the FCR calculations on 32 patients for vertebra L2 is shown in Fig. 2. Each vertebra is placed in the coordinate system of the vertebra below prior to the calculation of the FCR. The average scale factor was 0.9946 with a standard deviation of 0.0197 indicating that for these examples, the transformation between flexion and extension radiographs is very close to rigid motion. This is further verified by a very small mean residual error of 0.0775. Full clinical analysis of these results is beyond the scope of this technical note and will be published subsequently.

4. Discussion

In this technical note, we have shown a simpler derivation for calculating the angle of rotation of a set of marker points undergoing rigid planar rotation and translation. The major advantage of the approach, other than the simple derivation, is the automatic inclusion of the calculation of a scaling factor between the two point sets - the calculation of which was not obvious in previous approaches ([1,2]). The final numerical calculations are similar to those of [1] and are trivial to implement. A matlab routine for computing the two quantities, and the scaling factor, is included as Appendix A.

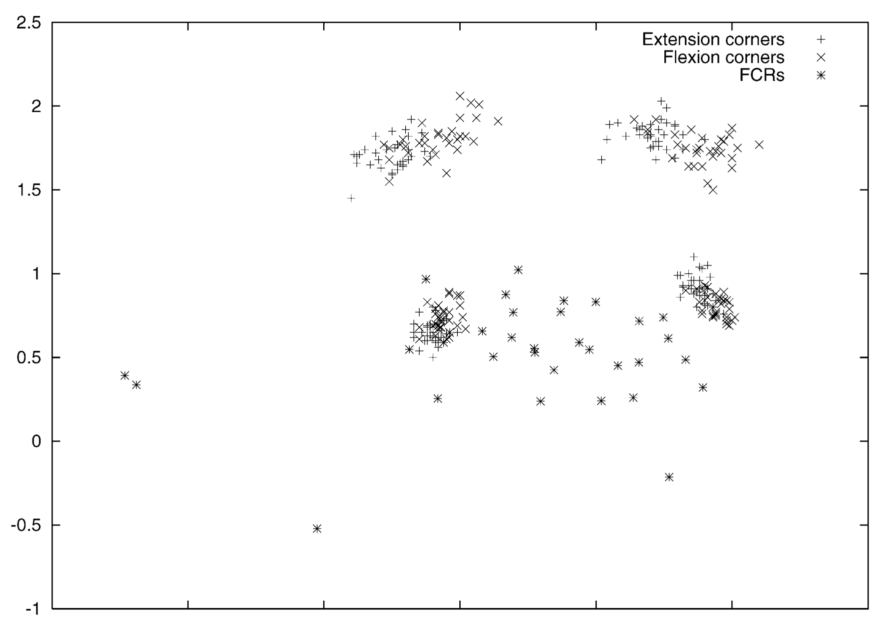

Fig. 2. FCR results for L2. The dimensions are unitless since they are normalised by the height of the vertebra below.

Appendix A. Matlab code

```
function [fcr, theta, scale] = calcFCRTheta(E, F)
% [fcr, theta, scale] = calcFCRTheta(E, F)
% calculate the finite centre of rotation, the angle of rotation,
% and the scale factor
% Input: E, F - lists of 2D corresponding points. These are n by 2 matrices
%               where n is the number of points and the x val is in column 1
%               y val in column 2.
% Output: fcr - the finite centre of rotation.
%       theta - the angle of rotation between the two sets of points
%       about the fcr.
%       scale - the scale factor between the two pointsets
% first do some error checking
[rE, cE] = size(E);
[rF, cF] = size(F);
if (cE ~= 2) | (cF ~= 2)
   error('E, F should be an nx2 matrix')
end
if (rE ~= rF)
   error('matrices E and F are of different size')
end
% complexify the data
A = E(:,1) + E(:,2)*i;
B = F(:,1) + F(:,2)*i;
% calculate the mean of each of the pointsets
meanA = sum(A)/rE;
meanB = sum(B)/rF;
% now translate both pointsets to the origin
A2 = A - meanA;
B2 = B - meanB;
% now find the least squares solution to the rotation between A and B
x = pinv(B2)*A2;
theta = angle(x);
scale = abs(x);
% the optimal rotation matrix
R = [cos(theta), -sin(theta); sin(theta), cos(theta)];
% the optimal translation vector
v = [real(meanA) imag(meanA)] - [real(meanB) imag(meanB)]*R';
% calculate the fcr
fcr(1,1) = (v(1,1)-v(1,2)*cot(theta/2))/2;
fcr(2,1) = (v(1,2)+v(1,1)*cot(theta/2))/2;
```

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