ON THE LENGTH OF THE LONGEST SUBSEQUENCE AVOIDING AN ARBITRARY PATTERN IN A RANDOM PERMUTATION

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Abstract. We consider the distribution of the length of the longest subsequence avoiding an arbitrary pattern, π, in a random permutation of length n. The well-studied case of a longest increasing subsequence corresponds to $\pi = 21$. We show that there is some constant c_π such that as $n \to \infty$ the mean value of this length is asymptotic to $2\sqrt{c_\pi n}$ and that the distribution of the length is tightly concentrated around its mean. We observe some apparent connections between c_π and the Stanley-Wilf limit of the class of permutations avoiding the pattern π.

Consider an arbitrary pattern avoidance class $\mathcal{A}$. Given any permutation π define the *longest* $\mathcal{A}$ *subsequences* of π or $\mathrm{L}\mathcal{A}\mathrm{S}(\pi)$ to be the set of those subsequences of π of maximum length, subject to the condition that their patterns belong to $\mathcal{A}$. Also define $L_{\mathcal{A}}(\pi)$ to be the length of any sequence in $\mathrm{L}\mathcal{A}\mathrm{S}(\pi)$. Let $\mathcal{I} = \mathrm{Av}(21)$ be the class of increasing permutations.

Apparently Ulam [10] was the first to ask the question:

> *What can be said about the distribution of values of* $L_{\mathcal{I}}(\Pi_n)$ *when* Π_n *is a random variable whose value is a permutation* π *chosen uniformly at random from among the elements of* $\mathcal{S}_n$*?*

We intend to address the generalization of this problem to the random variable $L_{\mathcal{A}}(\Pi_n)$ defined in a similar fashion. The history of the analysis of Ulam's problem is well documented in [2]. We repeat here a few details relevant to our investigations of the more general problem.

For convenience let $L_n = L_{\mathcal{I}}(\Pi_n)$. Ulam conjectured that for some constant c

$$\lim_{n\to\infty} \frac{\mathbf{E}\, L_n}{\sqrt{n}} = c.$$

This conjecture was proven by Hammersley [6] who showed also that $n^{-1/2} L_n \to c$ in probability, and who conjectured that $c = 2$. This further conjecture was proven in part by Logan and Shepp [8] and simultaneously in whole by Kerov and Veršik [11].

Frieze [5] and Bollobás and Brightwell [4] used martingale methods to establish tight concentration of L_n about its mean. Subsequently, Baik, Deift and Johansson [3] obtained complete asymptotic information about the distribution of L_n.

Our main theorems are analogs of the results of Hammersley and Frieze for the more general case of $L_{\mathcal{A}}(\Pi_n)$. For the first of these we need to impose a mild additional restriction on $\mathcal{A}$. The proofs for the general case are then essentially identical to the originals.

Theorem 1. *Let $\mathcal{A}$ be an infinite and proper pattern avoidance class which is either sum-closed or difference-closed. There exists a constant $1 \leq c_{\mathcal{A}} < \infty$ such that*

$$\lim_{n\to\infty} \frac{\mathbf{E}\, L_{\mathcal{A}}(\Pi_n)}{\sqrt{n}} = 2\sqrt{c_{\mathcal{A}}}. \tag{1}$$

Theorem 2. *Let $\mathcal{A}$ be a proper pattern class. For $\alpha > 1/3$ and $\beta < \min(\alpha, 3\alpha - 1)$*

$$\mathbf{Pr}\ (|L_{\mathcal{A}}(\Pi_n) - \mathbf{E}\, L_{\mathcal{A}}(\Pi_n)| \geq n^{\alpha}) < \exp(-n^{\beta}). \tag{2}$$

The Marcus-Tardos theorem is a key ingredient in the proof of Theorems 1 and 2. We will now present observations which provide some evidence for a connection between the constants $c_{\mathcal{A}}$ and

$$s_{\mathcal{A}} := \limsup_{n\to\infty} |\mathcal{S}_n \cap \mathcal{A}|^{1/n}.$$

Conjecture 1. For any proper pattern avoidance class $\mathcal{A}$, the limits superior definining $c_{\mathcal{A}}$ and $s_{\mathcal{A}}$ are in fact limits, and $c_{\mathcal{A}} = s_{\mathcal{A}}$.

The evidence for this conjecture is somewhat fragmentary at this point. It is supported by the following results.

Proposition 3. *For each positive integer k, the classes*

$$\mathrm{Av}(k(k-1)(k-2)\cdots 21) \quad \textit{and} \quad \mathrm{Av}(123\cdots k)$$

satisfy Conjecture 1.

Proposition 4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two pattern avoidance classes which satisfy Conjecture 1. Then their union, direct sum and juxtaposition also satisfy Conjecture 1. If, additionally, $\mathcal{A} \cap \mathcal{B}$ is a finite class then their merge also satisfies Conjecture 1.*

The results of the preceding proposition apply in part to the weaker version of Conjecture 1 which only asserts the equality of two limits superior.

Let $\mathcal{A}$ be any pattern avoidance class and define $\mathrm{Rot}(\mathcal{A})$ to be the set of permutations obtained by taking all the cyclic rotations of elements of $\mathcal{A}$. It is easily verified that $\mathrm{Rot}(\mathcal{A})$ is also a pattern avoidance class.

Proposition 5. *If $\mathcal{A}$ satisfies Conjecture 1 then so does* $\mathrm{Rot}(\mathcal{A})$.

Again, the "weak" form of this proposition is valid.

We have no significant evidence in favour of Conjecture 1 based on an exact computation of $c_{\mathcal{A}}$ for any classes other than those which can be produced from the classes $\mathrm{Av}(k(k-1)\cdots 321)$ or $\mathrm{Av}(123\cdots(m-1)m)$ by the constructions described in the previous section. An obvious starting point for the investigation of Conjecture 1 would be the collection of pattern avoidance classes whose basis consists entirely of permutations of length 3. These classes, as a group, were first analysed by Simion and Schmidt [9]. Starting from the "easy" end we note that almost all such classes having three or more basis elements satisfy Conjecture 1 as a consequence of Propositions 4 and 5 or trivial modifications of them. The exceptional cases are the classes whose growth is governed by the Fibonacci numbers, and we consider these below. Also, most of the cases having two basis elements are covered by Proposition 4. Up to symmetry there are two exceptions which we consider below.

- The *layered* permutations, $\mathcal{L} = \mathrm{Av}(231, 312)$, consisting of all permutations of the form $D_1 \oplus D_2 \oplus \cdots \oplus D_k$ where each of D_1 through D_k is a descending permutation. The number of permutations of length n in $\mathcal{L}$ is 2^{n-1} and so $s_{\mathcal{L}} = 2$.
- The subclass $\mathcal{L}(2) = \mathrm{Av}(231, 312, 321)$ of $\mathcal{L}$ formed by requiring that each D_i contain at most two elements. The number of permutations of length n in $\mathcal{L}(2)$ is equal to the nth Fibonacci number, so $s_{\mathcal{L}(2)} = (1 + \sqrt{5})/2$. Kaiser and Klazar [7] proved that $\mathcal{L}(2)$ is the smallest pattern avoidance class whose Stanley-Wilf limit is strictly greater than 1.
- The class $\mathcal{C} = \mathrm{Av}(321, 312)$ whose elements are those permutations that can be written as direct sums $C_1 \oplus C_2 \oplus \cdots \oplus C_k$ where each C_i is of the form $234\cdots n\,1$ for some $n > 1$, or simply 1.

Results in [1] give dynamic programming algorithms for solving the longest subsequence problem for both $\mathcal{L}$ and $\mathcal{L}(2)$ whose complexity is $O(n^2 \log n)$ where n is the length of the input permutation. We

Length	μ	σ	$\sim c_{\mathcal{L}(2)}$
1×10^4	239.3	4.5	1.431
2×10^4	340.7	5.2	1.451
4×10^4	484.7	6.1	1.468
8×10^4	688.4	6.4	1.481
16×10^4	978.1	7.1	1.495
32×10^4	1386.8	8.3	1.503
64×10^4	1965.3	9.3	1.510
128×10^4	2785.3	10.2	1.515

TABLE 1. Summary data for the mean, μ, and standard deviation σ of the length of the longest $\mathcal{L}(2)$ subsequences in a sample of 1000 random permutations together with corresponding estimates of $c_{\mathcal{L}(2)}$.

have been able to improve the latter algorithm, based on a tableau style method to result in a complexity of $O(n \log n)$. The algorithm provided in [1] for $\mathcal{C}$ has worst case complexity $O(n^3 \log n)$ though in practice with some minor optimisations it performs significantly better than this on random permutations. All three algorithms were implemented and a long period random number generator was used to provide experimental data concerning the values $c_{\mathcal{L}}$, $c_{\mathcal{L}(2)}$, and $c_{\mathcal{C}}$.

For $\mathcal{L}(2)$ we present data based on permutations of length $2^k \times 10^4$ for $0 \leq k \leq 7$. For each value of k, 1000 random permutations of that length were generated and the length of the longest $\mathcal{L}(2)$ subsequences was computed. Table 1 shows the mean, sample standard deviation, and resulting estimates of $c_{\mathcal{L}(2)}$ based on these simulations. We would be forced to classify a person who believed in the truth of Conjecture 1 based on this data for $\mathcal{L}(2)$ as an optimist. If the estimates are indeed converging to $s_{\mathcal{L}(2)}$ then they are not yet within 6% of their final limit at $n = 128 \times 10^4$. By contrast, for this value of n the estimate for $c_{\mathcal{I}}$ (whose actual value is 1) is approximately 0.985.

Because of the slower running time and increased space requirements required by the algorithm for finding longest layered subsequences data for $\mathcal{L}$ is based on permutations of length $2^k \times 10^2$ for $0 \leq k \leq 7$. As for $\mathcal{L}(2)$, 1000 random permutations of each length were analysed and the results are presented in Table 2. The data for this class do not require as much optimism as the $\mathcal{L}(2)$ data to be viewed as support for Conjecture 1.

Length	μ	σ	$\sim c_{\mathcal{L}}$
1×10^2	23.8	1.8	1.418
2×10^2	34.8	2.2	1.517
4×10^2	50.6	2.5	1.602
8×10^2	73.4	3.0	1.682
16×10^2	105.2	3.3	1.730
32×10^2	150.7	4.0	1.774
64×10^2	215.9	4.4	1.821
128×10^2	307.5	4.9	1.847

TABLE 2. Summary data for the mean, μ, and standard deviation σ of the length of the longest $\mathcal{L}$ subsequences in a sample of 1000 random permutations together with corresponding estimates of $c_{\mathcal{L}}$.

Length	μ	σ	$\sim c_{\mathcal{L}}$
1×10^2	22.9	2.0	1.306
2×10^2	33.5	2.3	1.406
4×10^2	48.5	2.4	1.470
8×10^2	70.5	3.1	1.555
16×10^2	101.2	3.3	1.601
32×10^2	145.2	3.9	1.647

TABLE 3. Summary data for the mean, μ, and standard deviation σ of the length of the longest $\mathcal{C}$ subsequences in a sample of 1000 random permutations together with corresponding estimates of $c_{\mathcal{C}}$.

Finally, the data for the class $\mathcal{C}$ presented in Table 3 is even more limited, but again it seems to provide qualified support for Conjecture 1.

Notable by its omission from our discussion is the class Av(312). This class has Stanley-Wilf limit 4. A polynomial time algorithm for the longest subsequence problem based on this class is given in [1] but its complexity on permutations of length n is $O(n^5)$ which makes it impractical for experiments of the size required to produce even vaguely convincing evidence. The goal of producing such evidence would seem to require finding, even on an *ad hoc* basis some collection of classes for which the longest subsequence problem can be solved algorithmically in reasonable time (basically, at worst quadratic) and/or developing better algorithms for classes such as Av(312).

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