Permutation classes of polynomial growth

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1 Introduction

A permutation π is said to be a subpermutation of a permutation σ if σ has a subsequence isomorphic to π (that is, its terms are ordered relatively the same as the terms of π). For example 312 is a subpermutation of 25134 because of the subsequence 513 (or 514 or 534). On the other hand 321 is not a subpermutation of 25134 because there is no three element subsequence of 25134 in which the three elements occur in decreasing order. Consequently 25134 is said to *involve* 312 but to *avoid* 321. If Π is a set of permutations then $\mathrm{Av}(\Pi)$ denotes the set of all permutations which avoid every permutation in Π. Such sets of permutations are called *pattern classes* and have given rise to many enumerative results. Typically, given Π, one is interested in determining the number $c_n(\Pi)$ of permutations of each length n in the pattern class $\mathrm{Av}(\Pi)$. For obvious reasons we shall assume throughout that Π is non empty. When explicitly listing the elements of some set Π as an argument we will generally omit braces, thus writing $c_n(123, 312)$ rather than $c_n(\{123, 312\})$.

The sequences $c_n(\Pi)$ can be studied from several points of view. We might wish to discover an exact formula for $c_n(\Pi)$, to find bounds on its growth as a function of n, or to determine the ordinary generating function

$$\sum_{\sigma \in \mathrm{Av}(\Pi)} x^{|\sigma|}.$$

Recently Marcus and Tardos [8] resolved affirmatively the long-standing open question of whether $c_n(\Pi)$ was always exponentially bounded. In part because

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of this result, attention has turned to enumerative questions of finer detail, and this paper addresses one such question.

We shall be concerned with pattern classes of polynomial growth; in other words, classes $\mathrm{Av}(\Pi)$ for which there exists a bound of the form

$$c_n(\Pi) \leq An^d$$

for some constants A, d. Such classes were studied by Kaiser and Klazar in [7]; they proved that, in such a class, $c_n(\Pi)$, as a function of n, was actually equal to some polynomial for all sufficiently large n and that this polynomial had a particular form. Kaiser and Klazar also proved that classes $\mathrm{Av}(\Pi)$ whose growth was not polynomial have $c_n(\Pi) \geq \tau^n$ where τ is the golden ratio.

Examples of classes of polynomial growth have also appeared many times in the literature. For example, in an early paper [10], on pattern class enumeration, Simion and Schmidt proved that $c_n(132, 321) = n(n-1)/2 + 1$. Some more difficult enumerations were carried out by West [11] in his work on classes of the form $\mathrm{Av}(\alpha, \beta)$ where α is a permutation of length three, and β one of length four; he showed that 4 of the 18 essentially different such classes have polynomial enumerations.

In the next section we investigate the necessary and sufficient conditions on Π for $\mathrm{Av}(\Pi)$ to have polynomial growth. These conditions (Theorem 1) turn out to be so simple that it is virtually trivial to test whether $\mathrm{Av}(\Pi)$ has polynomial growth. Some parts of this result are already implicit in [7] but our approach is somewhat different. In particular we focus on the structural characteristics of the elements of such permutation classes and are able to provide a uniform argument leading to the desired conditions. By themselves the conditions tell us little about an actual polynomial that gives $c_n(\Pi)$ (for sufficiently large n) and so, in Section 3, we go on to give more precise results when $|\Pi| \leq 3$. Huczynska and Vatter have also recently provided in [6] an alternative simplification of the proof of the result of Kaiser and Klazar, establishing the dichotomy between classes of polynomial growth and those whose growth exceeds the growth of the Fibonacci numbers.

If $\Pi = \{\alpha\}$ there is nothing to say beyond what is obvious; $c_n(\alpha)$ has polynomial growth only if $|\alpha| \leq 2$. In these cases:

$$\begin{array}{rcll} c_n(1) & = & 0 & \text{for all } n \geq 1 \\ c_n(12) = c_n(21) & = & 1 & \text{for all } n \geq 1 \end{array}$$

If Π has two or three elements the conditions for polynomial growth are more complex. In the latter case the classes $\mathrm{Av}(\Pi)$ of polynomial growth are sufficiently numerous that we have only used Theorem 1 to list the various sets Π (see Theorem 4); it would not be difficult in most cases to give the complete enumerations. However, in the former case, we obtain a characterisation (Theorem 3) of polynomial growth classes which are more demanding to analyse. In

Section 4 we give some bounds on the degrees of the polynomials that arise in this case.

In order to simplify the exposition it will be useful to introduce a few further pieces of definition and notation. Two sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ of distinct elements from (possibly different) totally ordered sets are *order isomorphic* (or simply *equivalent*) if, for all $1 \leq i, j \leq n$, $a_i < a_j$ if and only if $b_i < b_j$. Thus, a permutation π is involved in a permutation σ exactly when, considered as a sequence, it is equivalent to some subsequence of σ. Further, every finite sequence of distinct elements from a totally ordered set is equivalent to exactly one permutation, called its *pattern*. If a pattern class $X = \mathrm{Av}(\Pi)$ is of polynomial growth, then we define degree(X) to be the degree of the polynomial p for which $c_n(\Pi) = p(n)$ for all sufficiently large n.

2 Conditions for polynomial growth

The main result of this section is a necessary and sufficient condition that $\mathrm{Av}(\Pi)$ has polynomial growth. Informally the condition is that among the permutations of Π we must find permutations of all the 10 types shown in Figure 1. Clearly, testing this condition is very easy.

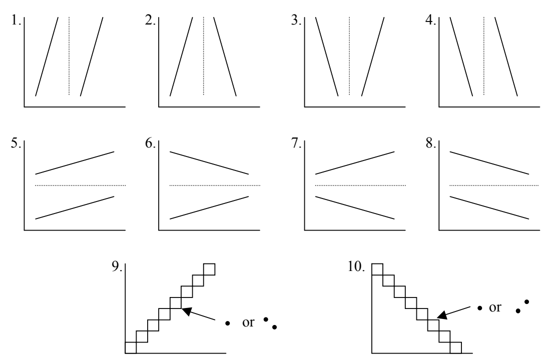

Figure 1: 10 types of permutation

We shall develop some terminology and notation to state this condition more formally, and to justify it. Let $\epsilon = (e_1, e_2, \ldots, e_r)$ be any sequence whose terms are $+1$ or -1. Then the pattern class $W(\epsilon)$ consists of all permutations π that have a segmentation

$$\pi = \sigma_1 \sigma_2 \cdots \sigma_r$$

where σ_i is increasing if $e_i = +1$ and decreasing if $e_i = -1$. These pattern classes are the 'W'-classes of [3, 1] where they were used to study partial well-order and regularity questions. We will be particularly interested in the four

W-classes formed from sequences ϵ of length 2 and their inverses, which form the first eight types in Figure 1. We will use a somewhat more compact notation for these classes:

$$W_{+-} = W(+1, -1) \quad W_{--}^{-1} = W(-1, -1)^{-1} \quad \text{etc.}$$

The last two types in Figure 1 are not related to W-classes and we call them L_2 and L_2^R respectively since the first consists of permutations with increasing layers which are either singletons or decreasing doubletons, and the second is the reverse of this class. We can now state our condition formally:

Theorem 1 *A pattern class $Av(\Pi)$ has polynomial growth if and only if every class in the list*

$$\begin{gathered} W_{++},\, W_{+-},\, W_{-+},\, W_{--} \\ W_{++}^{-1},\, W_{+-}^{-1},\, W_{-+}^{-1},\, W_{--}^{-1} \\ L_2,\, L_2^R \end{gathered}$$

has non-empty intersection with Π.

The method of proof shows that every class of polynomial growth is a subclass of a polynomial growth class $P(\pi, \mu)$ defined by a permutation π (of degree m say) and a sequence μ of m associated signs ± 1. The permutations in the class $P(\pi, \mu)$ are obtained from π by replacing any term associated with $+1$ by an increasing consecutive segment (possibly empty), and the terms associated with -1 by a decreasing consecutive segment. Therefore any permutation in the class can be specified (though not generally uniquely) by the vector of lengths of these segments. A subclass then corresponds to an ideal in the partially ordered set of such vectors ordered by dominance.

Furthermore the proof shows also that:

Corollary 2 *Every pattern class of polynomial growth is finitely based.*

3 Two or three restrictions

Theorem 1 does not give any hint about what the degree of a class given as $\mathrm{Av}(\Pi)$ is. In this section we consider the implications of Theorem 1 for Π when $|\Pi| = 2$ or 3. To eliminate trivialities we will assume throughout this section that each permutation in Π has length at least three. The methods of proof are patient enumerations of cases and we omit them in this abstract.

Theorem 3 *Let $X = Av(\alpha, \beta)$ have polynomial growth. Then, up to symmetry and exchange of α with β, we have one of the following:*

1. *α is increasing and β is decreasing,*

2. *α is increasing and β is almost decreasing in the sense that $\beta \in L_2^R$ with exactly one layer of size 2.*

We find it illuminating to contrast this result with the Erdös-Szekeres theorem which can be recast in the form: a pattern class $\mathrm{Av}(\alpha, \beta)$ is finite if and only if one of α and β is increasing and the other is decreasing.

Theorem 4 *Let $Av(\alpha, \beta, \gamma)$ have polynomial growth. Then, up to symmetry and exchange of α with β, we have one of the following:*

1. *$\alpha = 213$, and*
   (a) *$3412 \preceq \beta \in L_2^R$ and $\gamma = 12 \cdots k\, n(n-1) \cdots (k+1)$ for some k, or*
   (b) *$\beta = m(m-1) \cdots (j+2)\, j(j+1)\, (j-1)(j-2) \cdots 1$ and $\gamma = 12 \cdots k\, n(n-1) \cdots (k+1)$ for some j, k, or*
   (c) *$\beta = m(m-1) \cdots 312$ and $\gamma \in W_{+-}$.*
2. *α is increasing, $3412 \preceq \beta \in L_2^R$ and $\gamma \in W_{--} \cap W_{--}^{-1}$, (or $Av(\alpha, \beta)$ has polynomial growth).*
3. *$\alpha = 21345 \cdots s$, and*
   (a) *$3412 \preceq \beta \in L_2^R$ and $\gamma = 1n(n-1) \cdots 2$, or*
   (b) *$\beta = m(m-1) \cdots (j+2)\, j(j+1)\, (j-1)(j-2) \cdots 1$ and $\gamma = 12 \cdots k\, n(n-1) \cdots (k+1)$ for some j, k, or*
   (c) *$\beta = m(m-1) \cdots 312$ and $\gamma \in W_{+-}$.*

4 Enumeration when there are two restrictions

The theorems above provide no information on how to find the degrees of the polynomials that enumerate classes of polynomial growth. We make a start on this problem for classes with two restrictions only and throughout this section we shall consider classes $\mathrm{Av}(\alpha, \beta)$ defined by two restrictions of the form given in Theorem 3. Specifically, for some positive integer r and non-negative integers p and q:

1. $\alpha = \alpha_r = 12 \cdots r$, and
2. $\beta = \beta_{pq} = \lambda\, (q+1)\, (q+2)\, \mu$ where $|\lambda| = p, |\mu| = q$, λ is decreasing with consecutive terms all of which are greater than $q+2$, and μ is decreasing with consecutive terms, all of which are less than $q+1$. Define $s = |\beta| = p + q + 2$.

We shall give upper and lower bounds on $\mathrm{degree}(\mathrm{Av}(\alpha_r, \beta_{pq}))$ for arbitrary r, p, q, and some tighter bounds in small special cases. Our results are as follows:

Theorem 5

$$(r-1)s-2)-1 \leq \mathit{degree}(Av(\alpha_r, \beta_{pq})) \leq \begin{cases} (r-1)^2(s-2)-r & \text{if } p > 0 \text{ and } q > 0, \\ (r-1)^2(s-2)-1 & \text{if } p = 0 \text{ or } q = 0. \end{cases}$$

Theorem 6 *The pattern class $Av(12\cdots r, 231)$ is enumerated by a polynomial of degree $2r-3$ with leading coefficient* $\mathrm{cat}(r-2)$.

Theorem 7 *If $p > 0$ and $q > 0$ then degree$(Av(123, \beta_{pq})) = 2s-3$. If either $p = 0$ or $q = 0$, degree$(Av(123, \beta_{pq})) = 2s-4$.*

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