Simple permutations

Robert Brignall
joint work with Sophie Huczynska*, Nik Ruškuc, and Vincent Vatter†
School of Mathematics and Statistics
University of St Andrews
St Andrews, Fife, Scotland
{robertb, sophieh, nik, vince}@mcs.st-and.ac.uk
http://turnbull.mcs.st-and.ac.uk/~{robertb, sophieh, nik, vince}

An *interval* in the permutation π is a set of contiguous indices $I = [a, b]$ such that the set of values $\pi(I) = \{\pi(i) : i \in I\}$ also forms an interval of natural numbers. Every permutation π of $[n] = \{1, 2, \ldots, n\}$ has intervals of size 0, 1, and n; π is said to be *simple* if it has no other intervals. The series of papers [3, 4, 5] harnesses simple permutations to answer three related, but very different, questions.

Algebraic generating functions for sets of permutations.

Substitution decompositions (known in other contexts as modular decompositions, disjunctive decompositions and X-joins) have proved to be a useful technique in a wide range of settings, ranging from game theory to combinatorial optimisation, see Möhring [7] or Möhring and Radermacher [8]. Although substitution decompositions are most often applied to algorithmic problems, in the study of permutation classes[1] they are applied to enumeration questions.

Albert and Atkinson [1] were the first to establish the link between simple permutations and the enumeration of permutation classes; they proved that every permutation class with only finitely many simple permutations has an algebraic generating function. In [3] we generalise their theorem to "finite query-complete sets of properties." Upon specialisation, this yields the following result.

Theorem 1. *In a permutation class $\mathcal{C}$ with only finitely many simple permutations, the following sequences have algebraic generating functions:*

- *the number of permutations in $\mathcal{C}_n$,*
- *the number of alternating permutations in $\mathcal{C}_n$,*

*Supported by a Royal Society Dorothy Hodgkin Research Fellowship.
†Supported by EPSRC grant GR/S53503/01.

[1] The permutation π is said to *contain* the permutation σ if π has a subsequence that is order isomorphic to σ. This pattern-containment relation is a partial order on permutations. We refer to downsets of permutations under this order as *permutation classes*. In other words, if $\mathcal{C}$ is a permutation class, $\pi \in \mathcal{C}$, and $\sigma \leq \pi$, then $\sigma \in \mathcal{C}$. We denote by $\mathcal{C}_n$ the set $\mathcal{C} \cap S_n$, i.e. those permutations in $\mathcal{C}$ of length n, and we refer to $\sum |\mathcal{C}_n| x^n$ as the generating function for $\mathcal{C}$.

- *the number of even permutations in $\mathcal{C}_n$,*
- *the number of Dumont permutations in $\mathcal{C}_n$,*
- *the number of permutations in $\mathcal{C}_n$ avoiding any finite set of blocked, barred, or boxed permutations, and*
- *the number of involutions in $\mathcal{C}_n$.*

Moreover, these conditions can be combined in any (finite) manner desired.

A decomposition theorem with enumerative consequences.

In [4] we prove that long simple permutations must contain two almost disjoint simple subsequences. Formally:

Theorem 2. *There is a function $g(k)$ such that every simple permutation of length at least $g(k)$ contains two simple subsequences, each of length at least k, which share at most two entries in common.*

Theorem 2 is motivated by a number of enumerative results for classes with only finitely many simple permutations. As we have already seen, such classes have algebraic generating functions. One class with only finitely many simple permutations is $\mathrm{Av}(132)$. Theorems 1 and 2 can be used to give a short proof of the following result.

Theorem 3 (Bóna [2]; Mansour and Vainshtein [6])**.** *For every fixed r, the class of all permutations containing at most r copies of 132 has an algebraic generating function*

Proof of Theorem 3 via Theorems 1 and 2. We wish to show that only finitely many simple permutations contain at most r copies of 132, or in other words, that there is a function $h(r)$ so that every simple permutation of length at least $h(r)$ contains more than r copies of 132. We have observed already that we may take $h(0) = 3$. We now proceed by induction, setting $h(r) = g(h(\lfloor r/2 \rfloor))$, where f is the function from Theorem 2. By that theorem, every simple permutation π of length at least $h(r)$ contains two simple subsequences of length at least $h(\lfloor r/2 \rfloor)$. By induction each of these simple subsequences contains more than $\lfloor r/2 \rfloor$ copies of 132. Moreover, because these simple susbequences share at most two entries in common, their copies of 132 are distinct, and thus π contains more than r copies of 132, as desired. □

In fact, our proof gives a stronger result, that every permutation class whose members contain a bounded number of copies of 132 has an algebraic generating function, whereas Theorem 3 concerns only the entire class of permutations with at most r copies of 132. Additionally, there is of course nothing special about 132. Denote by $\mathrm{Av}(\beta_1^{\leq r_1}, \beta_2^{\leq r_2}, \ldots, \beta_k^{\leq r_k})$ the class of permutations that have at most r_1 copies of β_1, at most r_2 copies of β_2, and so on. From this, we obtain the following result.

Corollary 4. *If the class $\mathrm{Av}(\beta_1, \beta_2, \ldots, \beta_k)$ contains only finitely many simple permutations then for all choices of nonnegative integers r_1, r_2, $\ldots$, and r_k, the class $\mathrm{Av}(\beta_1^{\leq r_1}, \beta_2^{\leq r_2}, \ldots, \beta_k^{\leq r_k})$ also contains only finitely many simple permutations.*

The largest permutation class whose only simple permutations are 1, 12 and 21 is the class of *seperable permutations*, $\mathrm{Av}(2413, 3142)$. Thus as an example of Corollary 4, we have the following result.

Corollary 5. *For all r and s, every subclass of $\mathrm{Av}(2413^{\leq r}, 3142^{\leq s})$ contains only finitely many simple permutations and thus has an algebraic generating function.*

Decidability and unavoidable structures.

We have already seen that being able to tell whether a given permutation class has finitely many simple permutations or not is important for enumeration, and so it is natural to ask whether this property of a class is decidable, and, if so, how to determine this property. We answer this in [5].

Theorem 6. *It is decidable whether a finitely based permutation class contains only finitely many simple permutations.*

The proof of this result is constructive, so can be used directly to construct an algorithm answering this question for any specified permutation class.

We also prove in [5] an unavoidable structure result about simple permutations, namely:

Theorem 7. *Every sufficiently long simple permutation contains an "alternation" or "oscillation" of length k.*

References

[1] ALBERT, M. H., AND ATKINSON, M. D. Simple permutations and pattern restricted permutations. *Discrete Math. 300*, 1-3 (2005), 1–15.

[2] BÓNA, M. The number of permutations with exactly r 132-subsequences is P-recursive in the size! *Adv. in Appl. Math. 18*, 4 (1997), 510–522.

[3] BRIGNALL, R., HUCZYNSKA, S., AND VATTER, V. Simple permutations I: algebraic generating functions for sets of permutations. In preparation.

[4] BRIGNALL, R., HUCZYNSKA, S., AND VATTER, V. Simple permutations II: a decomposition theorem and its enumerative consequences. In preparation.

[5] BRIGNALL, R., RUŠKUC, N., AND VATTER, V. Simple permutations III: decidability and unavoidable substructures. In preparation.

[6] MANSOUR, T., AND VAINSHTEIN, A. Counting occurrences of 132 in a permutation. *Adv. in Appl. Math. 28*, 2 (2002), 185–195.

[7] MÖHRING, R. H. Algorithmic aspects of the substitution decomposition in optimization over relations, sets systems and Boolean functions. *Ann. Oper. Res. 4*, 1-4 (1985), 195–225.

[8] MÖHRING, R. H., AND RADERMACHER, F. J. Substitution decomposition for discrete structures and connections with combinatorial optimization. In *Algebraic and combinatorial methods in operations research*, vol. 95 of *North-Holland Math. Stud.* North-Holland, Amsterdam, 1984, pp. 257–355.