Extended abstract for talk in Permutation Patterns 2006

Title of paper in preparation: A note on log-convexity of q-Catalan numbers
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Requested date of presentation: June 16, 2006

The k^{th} Catalan number C_k is the number of lattice permutations with k 1s and k 2s. So, if the permutation is $\pi = \pi_1\pi_2\cdots\pi_{2k}$ then the number of 1s in $\pi_1\pi_2\cdots\pi_i$ is at least the number of 2s in $\pi_1\pi_2\cdots\pi_i$ for $1 \leq i \leq 2k$. A lattice permutaion may be visualized as a path in the lattice $Z \times Z$, with n steps (1,0) and n steps (0,1), such that the path never rises above the diagonal that joins its two endpoints. Lattice permutations were studied by MacMahon[3]. The sequence of Catalan numbers, C_k for $k \geq 1$,

$$1, 2, 5, 14, \ldots$$

is log-convex, a fact that is easily deduced from the formula $C_k = \frac{1}{k+1}\binom{2k}{k}$. That is, $C_{k-1}C_{k+1} \geq C_k^2$ for $k \geq 2$. For example, $C_2C_4-(C_3)^2 = 2\cdot 14-5^2 = 3$ is nonnegative.

Many polynomials in q have been called q-Catalan numbers. See [2]. Our work focuses on the q-Catalan numbers defined using inversion number of the multiset permutation; the inversion number is the area underneath the lattice path, which joins opposite vertices in a $k \times k$ square. So $C_k(q) = \sum_\pi q^{\operatorname{inv}\pi}$ is a polynomial in q with nonnegative integer coefficients of degree $\binom{k}{2}$. The sequence of q-Catalan numbers, $C_k(q)$ for $k \geq 1$,

$$1, 1+q, 1+q+2q^2+q^3, 1+q+2q^2+3q^3+3q^4+3q^5+q^6, \ldots$$

is not q-log-convex, since there are negative coefficients in $C_2(q)C_4(q) - (C_3(q))^2$. However, we prove that $C_{k-1}(q)C_{k+1}(q) - q(C_k(q))^2$ has nonnegative coefficients for $k \geq 2$. For example, $C_2(q)C_4(q) - q(C_3(q))^2 = 1+q+q^2$ has nonnegative coefficients. Our main result is the following:

Theorem: The q-Catalan numbers $C_n(q) = \sum_\pi q^{\operatorname{inv}\pi}$, where the sum is over lattice permutations with n 1s and n 2s, satisfy:

$$C_{k-1}(q)C_{\ell+1}(q) - q^{\ell-k+1}C_k(q)C_\ell(q) \text{ has nonnegative coefficients for } k \leq \ell.$$

The proof of the above result uses an injection introduced by Butler[1] to prove a similar log-concavity result for q-binomial coefficients:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ \ell \end{bmatrix}_q - q^{\ell-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \begin{bmatrix} n \\ \ell+1 \end{bmatrix}_q \text{ has nonnegative coefficients for } k \le \ell.$$

Proof of the Theorem: Given $k \le \ell$, we define an injection

$$\varphi : \mathcal{P}_k \times \mathcal{P}_\ell \to \mathcal{P}_{k-1} \times \mathcal{P}_{\ell+1}$$

where $\mathcal{P}_n$ is the set of lattice permutations with n 1s and n 2s. Then we show that if $\varphi(\pi, \sigma) = (\nu, \omega)$, we have $\operatorname{inv} \pi + \operatorname{inv} \sigma + (\ell - k + 1) = \operatorname{inv} \nu + \operatorname{inv} \omega$.

Let $(\pi, \sigma) \in \mathcal{P}_k \times \mathcal{P}_\ell$ be given. For $0 \le i \le 2k-2$, consider

$$\begin{aligned} \nu^{(i)} &= \sigma_1 \cdots \sigma_i \pi_{i+3} \cdots \pi_{2k} \\ \omega^{(i)} &= \pi_1 \cdots \pi_i \pi_{i+1} \pi_{i+2} \sigma_{i+1} \cdots \sigma_{2\ell} \end{aligned}$$

Define $\varphi(\pi, \sigma) = (\nu^{(i)}, \omega^{(i)}) = (\sigma_L \pi_R, \pi_L \sigma_R)$, where i is smallest such that the number of 2s in π_L exceeds the number of 2s in σ_L by exactly 1. (Note that $\pi_L = \pi_1 \pi_2$ and $\sigma_L = \emptyset$ if $i = 0$, but $\pi_L = \pi$ and $\sigma_L = \sigma_1 \cdots \sigma_{2k-2}$ if $i = 2k-2$. Initially π_L has at most one more 2 than σ_L, and finally π_L has at least one more 2 than σ_L.)

To show that $\operatorname{inv} \pi + \operatorname{inv} \sigma + (\ell - k + 1) = \operatorname{inv} \sigma_L \pi_R + \operatorname{inv} \pi_L \sigma_R$, we use the fact that the number of 2s in π_L, denoted $m_2 \pi_L$, equals $m_2 \sigma_L + 1$; hence the number of 1s in π_L, denoted $m_1 \pi_L$, equals $m_1 \sigma_L + 1$. Since an inversion 21 in a permutation may occur in the left portion, occur in the right portion, or straddle the left and right portions, we have:

$$\begin{aligned} \operatorname{inv} \pi &= \operatorname{inv} \pi_L + \operatorname{inv} \pi_R + (m_2 \pi_L)(m_1 \pi_R) \\ \operatorname{inv} \sigma &= \operatorname{inv} \sigma_L + \operatorname{inv} \sigma_R + (m_2 \sigma_L)(m_1 \sigma_R) \\ \operatorname{inv} \pi_L \sigma_R &= \operatorname{inv} \pi_L + \operatorname{inv} \sigma_R + (m_2 \pi_L)(m_1 \sigma_R) \\ \operatorname{inv} \sigma_L \pi_R &= \operatorname{inv} \sigma_L + \operatorname{inv} \pi_R + (m_2 \sigma_L)(m_1 \pi_R) \end{aligned}$$

$$\begin{aligned} \operatorname{inv} \pi_L \sigma_R + \operatorname{inv} \sigma_L \pi_R - \operatorname{inv} \pi - \operatorname{inv} \sigma &= (m_2 \pi_L - m_2 \sigma_L)(m_1 \sigma_R - m_1 \pi_R) \\ &= m_1 \sigma_R - m_1 \pi_R \\ &= (\ell - m_1 \sigma_L) - (k - m_1 \pi_L) \\ &= \ell - k + 1 \end{aligned}$$

Corollary: For $1 \leq r \leq k$ and $\ell > k - r$,

$$C_{k-r}(q)C_{\ell+r}(q) - q^{r(\ell-k+r)}C_k(q)C_\ell(q) \text{ has nonnegative coefficients.}$$

The Corollary may be proved by induction on r or may be proved using an injection like the one above for $r = 1$. To visualize this injection φ, picture π as a lattice path L_π from the lower left corner to the upper right corner of a $k \times k$ square and picture σ as a lattice path L_σ from the lower left corner to the upper right corner of a $\ell \times \ell$ square. Arrange these inside a $(\ell+r) \times (\ell+r)$ square with the lattice path L_π beginning in the lower left corner and the lattice path L_σ ending in the upper right corner. Find the point where L_π first meets L_σ. If $\varphi(\pi, \sigma) = (\nu, \omega)$, then the lattice path L_ω follows L_π until this point then follows L_σ, and the lattice path L_ν follows L_σ until this point then follows L_π. The exponent $r(\ell - k + r)$ is the area of the rectangle in the lower right of the $(\ell + r) \times (\ell + r)$ square that lies to the right of the $k \times k$ square in the lower left and lies below the $\ell \times \ell$ square in the upper right.

References:

[1] L. M. Butler, "The q-log-concavity of q-binomial coefficients", *J. Combin. Theory Ser. A* **54** (1990), 54–63.

[2] J. Fürlinger and J. Hofbauer, "q-Catalan numbers", *J. Combin. Theory Ser. A* **40** (1985), 248–264.

[3] P. A. MacMahon, *Combinatory Analysis*, Vol. I, Cambridge, 1915.