GENERATING TREES FOR PERMUTATIONS AVOIDING GENERALIZED PATTERNS

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Abstract. We construct generating trees with with one and two labels for some classes of permutations avoiding generalized patterns of length 3 and 4. These trees are built by adding at each level an entry to the right end of the permutation, instead of inserting always the largest entry. This allows us to incorporate the adjacency condition about some entries in an occurrence of a generalized pattern. We find functional equations for the generating functions enumerating these classes of permutations with respect to different parameters, and in a few cases we solve them using some techniques of Bousquet-Mélou [1], recovering known enumerative results and finding new ones.

1. Introduction

1.1. **Generalized pattern avoidance.** We denote by $\mathcal{S}_n$ the symmetric group on $\{1, 2, \ldots, n\}$. Let n and k be two positive integers with $k \leq n$, and let $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathcal{S}_n$ be a permutation. A generalized pattern σ is obtained from a permutation $\sigma_1\sigma_2 \cdots \sigma_k \in \mathcal{S}_k$ by choosing, for each $j = 1, \ldots, k-1$, either to insert a dash - between σ_j and σ_{j+1} or not. More formally, $\sigma = \sigma_1\varepsilon_1\sigma_2\varepsilon_2 \cdots \varepsilon_{k-1}\sigma_k$, where each ε_j is either the symbol - or the empty string. With this notation, we say that π *contains* (the generalized pattern) σ if there exist indices $i_1 < i_2 < \ldots < i_k$ such that (i) for each $j = 1, \ldots, k-1$, if ε_j is empty then $i_{j+1} = i_j + 1$, and (ii) for every $a, b \in \{1, 2, \ldots, k\}$, $\pi_{i_a} < \pi_{i_b}$ if and only if $\sigma_a < \sigma_b$. In this case, $\pi_{i_1}\pi_{i_2} \cdots \pi_{i_k}$ is called an *occurrence* of σ in π.

If π does not contain σ, we say that π *avoids* σ, or that it is σ*-avoiding*. For example, the permutation $\pi = 3542716$ contains the pattern 12-4-3 because it has the subsequence 3576. On the other hand, π avoids the pattern 12-43. We denote by $\mathcal{S}_n(\sigma)$ the set of permutations in $\mathcal{S}_n$ that avoid σ. More generally, if $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$ is a collection of generalized patterns, we say that a permutation π is Σ-avoiding if it avoids all the patterns in Σ simultaneously. We denote by $\mathcal{S}_n(\Sigma)$ the set of Σ-avoiding permutations in $\mathcal{S}_n$.

1.2. **Generating trees.** Generating trees are a useful technique to enumerate classes of pattern-avoiding permutations (see for example [8]). The nodes at each level of the generating tree are indexed by permutations of a given length. It is common in the literature to define the children of a permutation π of length n to be those permutations that are obtained by inserting the entry $n+1$ in $\pi = \pi_1\pi_2 \cdots \pi_n$ in such a way that the new permutation is still in the class. In this paper we consider a variation of this definition. Here, the children of a permutation π of length n are obtained by appending an entry to the right of π, and shifting up by one all the entries in π that were greater than or equal to the new entry. For example, if the entry 3 is appended to the right of $\pi = 24135$, the child that we obtain is 251463. Adding the new entry to the right of the permutation makes these trees well-suited to enumerate permutations avoiding generalized patterns, as we will see in Section 2.

For some classes of permutations, a label can be associated to each node of the tree in such a way that the number of children of a permutation and their labels depend only on the label of the parent. For example, in the tree for 1-2-3-avoiding permutations, we can label each node π with $m = \min\{\pi_i : \exists j < i \text{ with } \pi_j < \pi_i\}$ (or $m = n+1$ if $\pi = n \cdots 21$). Then, the children of a permutation with label (m) have labels $(m+1), (2), (3), \ldots, (m)$, corresponding to the appended entry being $1, 2, 3, \ldots, m$, respectively. This succession rule, together with the fact that the root ($\pi = 1 \in \mathcal{S}_1$) has label (2), completely determines the tree. From this rule one can derive a functional equation for the generating function that enumerates the permutations by their length and the label of the corresponding node in the tree. For generating trees with one label, these equations are well understood and their solutions are algebraic series.

However, in other cases, one label is not enough to describe the generating tree in terms of a succession rule. In Section 3 we consider some classes of permutations avoiding generalized patterns where to describe the generating tree we need to assign two labels to each node. Generating trees with two labels were used in [1] to enumerate restricted permutations. In fact, the inspiration for the present paper and many of the ideas used come from Bousquet-Mélou's work. One difference is that here trees are constructed by adding at each level an entry to the right end of the permutation, which allows us to keep track of elements occurring in adjacent positions.

Given a permutation $\pi \in \mathcal{S}_n$, we will write $r(\pi) = \pi_n$ to denote the rightmost entry of π.

2. Generating trees with one label

2.1. $\{2\text{-}1\text{-}3, \overline{2}\text{-}31\}$-avoiding permutations. Recall that a permutation π is said to avoid the *barred* pattern $\overline{2}$-31 if every occurrence of 21 in π is part of an occurrence of 2-31; equivalently, for any index i such that $\pi_i > \pi_{i+1}$ there is an index $j < i$ such that $\pi_i > \pi_j > \pi_{i+1}$.

We use M_n to denote the n-th Motzkin number. Recall that $\sum_{n\geq 1} M_n t^n = \frac{1-t-2t^2-\sqrt{1-2t-3t^2}}{2t^2}$. The next result seems to be a new interpretation of the Motzkin numbers.

Theorem 2.1. *The number of* $\{2\text{-}1\text{-}3, \overline{2}\text{-}31\}$*-avoiding permutations of size* n *is* M_n*.*

Proof. Consider the generating tree for $\{2\text{-}1\text{-}3, \overline{2}\text{-}31\}$-avoiding permutations, where to go down a level, a new entry is appended to the right of the permutation. Labeling each permutation with its rightmost entry $r = r(\pi)$, this tree is described by the succession rule $(r) \longrightarrow (1)\ (2)\ \cdots\ (r-1)\ (r+1)$ and the fact that the root has label (1). Indeed, the new entry appended to the right of π cannot be greater than $\pi_n + 1$ in order for the new permutation to be 2-1-3-avoiding, and it cannot be π_n because then it would create an occurrence of 21 that is not part of an occurrence of 2-31.

Defining $D(t,u) = \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,\overline{2}\text{-}31)} u^{r(\pi)} t^n = \sum_{r\geq 1} D_r(t) u^r$, this succession rule gives the following equation for the generating function:

$$D(t,u) = tu + t\sum_{r\geq 1} D_r(t)(u + u^2 + \cdots + u^{r-1} + u^{r+1}) = tu + \frac{t}{u-1}[D(t,u) - uD(t,1)] + tuD(t,u),$$

which can be written as

$$\left(1 - \frac{t}{u-1} - tu\right) D(t,u) = tu - \frac{tu}{u-1} D(t,1). \tag{1}$$

Now we apply the Kernel method. The values of u as a function of t that cancel the term multiplying $D(t,u)$ on the left hand side are $u_0 = u_0(t) = \frac{1+t\pm\sqrt{1-2t-3t^2}}{2t}$, of which the one with the minus sign is a well-defined formal power series in t. Substituting $u = u_0$ in (1) gives

$$D(t,1) = u_0 - 1 = \frac{1-t-\sqrt{1-2t-3t^2}}{2t},$$

which is the generating function for the Motzkin numbers. $\square$

2.2. $\{2\text{-}1\text{-}3, \overline{2}^o\text{-}31\}$-avoiding permutations. Extending the notion of barred patterns, we say that a permutation π avoids the pattern $\overline{2}^o$-31 if every occurrence of 21 in π is part of an odd number of occurrences of 2-31; equivalently, for any index i such that $\pi_i > \pi_{i+1}$, the number of indices $j < i$ such that $\pi_i > \pi_j > \pi_{i+1}$ is odd.

Theorem 2.2. *The number of* $\{2\text{-}1\text{-}3, \overline{2}^o\text{-}31\}$*-avoiding permutations of size* n *is*

$$|\mathcal{S}_n(2\text{-}1\text{-}3, \overline{2}^o\text{-}31)| = \begin{cases} \frac{1}{2k+1}\binom{3k}{k} & \text{if } n = 2k, \\ \frac{1}{2k+1}\binom{3k+1}{k+1} & \text{if } n = 2k+1. \end{cases}$$

Proof. The generating tree for $\{2\text{-}1\text{-}3, \overline{2}^o\text{-}31\}$-avoiding permutations can be described by the succession rule $(r) \longrightarrow (r+1)\ (r-1)\ (r-3)\cdots$, that is, the labels of the children of a node labeled r are the numbers $1 \leq j \leq r+1$ such that $r - j$ is odd. Let $J(t,u) = \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,\overline{2}^o\text{-}31)} u^{r(\pi)} t^n = \sum_{r\geq 1} J_r(t) u^r$, and let $J^e(t,u) = \sum_{r \text{ even}} J_r(t) u^r$. After some work, the succession rule translates into the following functional equation:

$$\left(1 - \frac{tu^3}{u^2-1}\right) J(t,u) = tu - \frac{tu^2}{u^2-1} J(t,1) + \frac{tu(u-1)}{u^2-1} J^e(t,1). \tag{2}$$

The kernel $1 - \frac{tu^3}{u^2-1}$ as a function in the variable u has three zeroes, two of which are complex conjugates. Denote them by $u_1 = a(t) + b(t)i$ and $u_2 = \bar{u_1} = a(t) - b(t)i$. Adding the equations $0 = u_i^2 - 1 - u_i J(t,1) + (u_i - 1)J^e(t,1)$ for $i = 1, 2$, we get $a(t)J(t,1) = a(t)^2 - b(t)^2 - 1 + (a(t)-1)J^e(t,1)$, and subtracting them gives

$J(t,1) = 2a(t) + J^e(t,1)$. Solving these two equations for J, we get that $J(t,1) = 2a(t) - a(t)^2 - b(t)^2 - 1$. Plugging the values of $a(t)$ and $b(t)$ yields the expression

$$J(t,1) = \frac{(2-3t)f(t)^2 + (9t-2-g(t))f(t) + (2-6t)g(t) + 54t^2 - 18t - 4}{3tf(t)^2},$$

where $g(t) = \sqrt{3(27t^2-4)}$ and $f(t) = [12tg(t) - 108t^2 + 8]^{1/3}$. It is easy to check that $J = J(t,1)$ satisfies $tJ^3 + (3t-2)J^2 + (3t-1)J + t = 0$, so its coefficients are given by the sequence A047749 from the on-line Encyclopedia of Integer Sequences [7]. Observe that we can also obtain an expression for $J(t,u)$ using (2) and the fact that $J^e(t,1) = -a(t)^2 - b(t)^2 - 1$. □

Analogously to the definition for the pattern $\overline{2}^o$-31, we say that a permutation π avoids the pattern $\overline{2}^e$-31 if every occurrence of 21 in π is part of an even number of occurrences of 2-31. An argument similar to the proof of Theorem 2.2 allows us to enumerate $\{$2-1-3$, \overline{2}^e$-31$\}$-avoiding permutations. Let $E(t) = \sum_{n\geq 1} |\mathcal{S}_n(\text{2-1-3}, \overline{2}^e\text{-31})|\, t^n$.

Theorem 2.3. *The generating function for* $\{$2-1-3$, \overline{2}^e$-31$\}$*-avoiding permutations is*

$$E(t) = \frac{(2-4t)\tilde{f}(t)^2 + (-2+12t-7t^2-\tilde{g}(t))\tilde{f}(t) + (2-8t)\tilde{g}(t) + 8t^3 + 46t^2 - 8t - 4}{3t\tilde{f}(t)^2},$$

where $\tilde{g}(t) = \sqrt{3(-5t^4 + 24t^3 - 4t^2 + 12t - 4)}$ *and* $\tilde{f}(t) = [4(3t\tilde{g}(t) - 11t^3 - 12t^2 - 6t + 2)]^{1/3}$.

The first coefficients of this generating function are $1, 2, 4, 9, 22, 56, 147, 396, \ldots$

2.3. **$\{$2-1-3, 2-3-41, 3-2-41$\}$-avoiding permutations.** Let $K(t,u) = \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(\text{2-1-3,2-3-41,3-2-41})} u^{r(\pi)} t^n = \sum_{r\geq 1} K_r(t) u^r$. The following result, combined with [6, Example 2.6], shows that $|\mathcal{S}_n(\text{2-1-3, 2-3-41, 3-2-41})| = |\mathcal{S}_n(\text{1-3-2, 123-4})|$.

Theorem 2.4. *The generating function for* $\{$2-1-3, 2-3-41, 3-2-41$\}$*-avoiding permutations with respect to the value of the rightmost entry is*

$$K(t,u) = \frac{1 - t - 2tu - \sqrt{1-2t-3t^2}}{2t(\frac{1}{u} + 1 + u) - 2}.$$

Proof. The succession rule for this class of permutations is

$$(r) \longrightarrow \begin{cases} (r-1)\ (r)\ (r+1) & \text{if } r > 1, \\ (r)\ (r+1) & \text{if } r = 1. \end{cases}$$

This translates into the functional equation

$$\left[1 - t\left(\frac{1}{u} + 1 + u\right)\right] K(t,u) = tu - tK_1(t). \tag{3}$$

Applying the Kernel method we find that $K_1(t) = \frac{1-t-\sqrt{1-2t-3t^2}}{2t}$, and substituting back into (3) we get the expression for $K(t,u)$. □

3. Generating trees with two labels

3.1. **$\{$2-1-3, 12-3$\}$-avoiding permutations.** It was shown in [3] that $|\mathcal{S}_n(\text{2-1-3, 12-3})| = M_n$. A bijection between $\mathcal{S}_n(\text{1-3-2, 1-23})$ and the set of Motzkin paths of length n was given in [4]. Clearly the sets $\mathcal{S}_n(\text{1-3-2, 1-23})$ and $\mathcal{S}_n(\text{2-1-3, 12-3})$ are equinumerous, since a permutation $\pi_1\pi_2\cdots\pi_n$ is $\{$1-3-2, 1-23$\}$-avoiding exactly when $(n+1-\pi_n)\cdots(n+1-\pi_2)(n+1-\pi_1)$ is $\{$2-1-3, 12-3$\}$-avoiding. In this section we recover the formula for $|\mathcal{S}_n(\text{2-1-3, 12-3})|$ using a generating tree with two labels. This method provides a refined enumeration of $\{$2-1-3, 12-3$\}$-avoiding permutations by to two new parameters: the value of the last entry and the smallest value of the top of an ascent.

Let $\mathcal{T}_1$ be the generating tree for the set of $\{$2-1-3, 12-3$\}$-avoiding permutations. Given any $\pi \in \mathcal{S}_n$, define the parameter

$$l(\pi) = \begin{cases} n+1 & \text{if } \pi = n(n-1)\cdots 21, \\ \min\{\pi_i : i > 1,\ \pi_{i-1} < \pi_i\} & \text{otherwise.} \end{cases} \tag{4}$$

Let the label of a permutation π be the pair $(l,r)=(l(\pi),r(\pi))$. Note that if π avoids $\{2\text{-}1\text{-}3, 12\text{-}3\}$, then necessarily $l \geq r$.

Proposition 3.1. *The generating tree $\mathcal{T}_1$ for $\{2\text{-}1\text{-}3, 12\text{-}3\}$-avoiding permutations is specified by the following succession rule on the labels:*

$$\begin{array}{l}(2,1)\\ (l,r) \longrightarrow \begin{cases} (l+1,1)\ (l+1,2)\ \cdots\ (l+1,l) & \text{if } l=r,\\ (l+1,1)\ (l+1,2)\ \cdots\ (l+1,r)\ (r+1,r+1) & \text{if } l>r.\end{cases}\end{array}$$

Proof. The permutation obtained by appending an entry to the right of $\pi \in \mathcal{S}_n(2\text{-}1\text{-}3, 12\text{-}3)$ is 2-1-3-avoiding if and only if the appended entry is at most $r(\pi)+1$, and it is 12-3-avoiding if and only if the appended entry is at most $l(\pi)$. The labels of the children are obtained by looking at how the values of (l,r) change when the new entry is added. □

We will use this description of the generating rule to obtain a formula for the generating function

$$M(t,u,v) := \sum_{n\geq 1} \sum_{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,12\text{-}3)} u^{l(\pi)} v^{r(\pi)}\ t^n.$$

For fixed l and r, let $M_{l,r}(t)=\sum_{n\geq1}|\{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,12\text{-}3): l(\pi)=l, r(\pi)=r\}|\ t^n$. Then we have that $M(t,u,v)=\sum_{l,r} M_{l,r}(t)u^l v^r$.

Theorem 3.2. *The generating function $M(t,u,v)$ enumerating $\{2\text{-}1\text{-}3, 12\text{-}3\}$-avoiding permutations with respect to size and the parameters l and r defined above can be expressed as*

$$M(t,u,v)=\frac{[(1-u)v+c_1t+c_2t^2+c_3t^3+c_4t^4-((1-u)v+tu+t^2u^2v)\sqrt{1-2t-3t^2})]u^2v}{2(1-u-tu(1-u)+t^2u^2)(1-uv+tuv+t^2u^2v^2)},$$

where $c_1=2-u-v-uv+2u^2v$, $c_2=u(-1+(2-u)v+2(u-1)v^2)$, $c_3=u^2v(-3+2v-2uv)$, and $c_4=-2u^3v^2$.

Notice that $M(t,1,1)$ is the generating function for the Motzkin numbers.

Proof. The coefficient of t^n in $M(t,u,v)$ is the sum of u^lv^r over all the labels (l,r) that appear at level n of the tree. By Proposition 3.1, the children of a node with label (l,l) contribute $u^{l+1}v+u^{l+1}v^2+\cdots+u^{l+1}v^l$ to the next level, and the children of a node with label (l,r) with $l>r$ contribute $u^{l+1}v+u^{l+1}v^2+\cdots+u^{l+1}v^r+u^{r+1}v^{r+1}$. It follows that

$$M(t,u,v)=tu^2v+t\sum_l M_{l,l}(t)u^{l+1}(v+v^2+\cdots+v^l)+t\sum_{l>r}M_{l,r}(t)[u^{l+1}(v+v^2+\cdots+v^r)+u^{r+1}v^{r+1}]. \tag{5}$$

It will be convenient to define

$$M_>(t,u,v):=\sum_{n\geq1}\sum_{\substack{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,12\text{-}3)\\ l(\pi)>r(\pi)}} u^{l(\pi)}v^{r(\pi)}\ t^n, \qquad M_=(t,u,v):=\sum_{n\geq1}\sum_{\substack{\pi\in\mathcal{S}_n(2\text{-}1\text{-}3,12\text{-}3)\\ l(\pi)=r(\pi)}} (uv)^{l(\pi)}\ t^n,$$

so that $M(t,u,v)=M_>(t,u,v)+M_=(t,u,v)$. Taking from (5) only the labels (l,r) with $l>r$, we get

$$\begin{aligned} M_>(t,u,v) &= tu^2v+t\sum_l M_{l,l}(t)u^{l+1}\frac{v^{l+1}-v}{v-1}+t\sum_{l>r}M_{l,r}(t)u^{l+1}\frac{v^{r+1}-v}{v-1}\\ &= tu^2v+\frac{tuv}{v-1}\left[M_=(t,u,v)-M_=(t,u,1)+M_>(t,u,v)-M_>(t,u,1)\right]. \end{aligned} \tag{6}$$

Similarly, taking from (5) only the labels (l,r) with $l=r$,

$$M_=(t,u,v)=t\sum_{l>r}M_{l,r}(t)u^{r+1}v^{r+1}=tuv\sum_{l>r}M_{l,r}(t)(uv)^r=tuv\ M_>(t,1,uv).$$

Using this expression in (6) and collecting the terms in $M_>(t,u,v)$, we have

$$\left(1-\frac{tuv}{v-1}\right)M_>(t,u,v)=tu^2v+\frac{tuv}{v-1}\left[tuv\ M_>(t,1,uv)-tu\ M_>(t,1,u)-M_>(t,u,1)\right], \tag{7}$$

and substituting $u=1$ and collecting the terms in $M_>(t,1,v)$,

$$\left(1-\frac{t^2v^2}{v-1}-\frac{tv}{v-1}\right)M_>(t,1,v)=tv-\frac{t(t+1)v}{v-1}M_>(t,1,1). \tag{8}$$

We apply the Kernel method, substituting $v=v_0=v_0(t)=\frac{1-t-\sqrt{1-2t-3t^2}}{2t^2}$ in (8) to obtain

$$M_>(t,1,1)=\frac{v_0-1}{t+1}=\frac{1-t-2t^2-\sqrt{1-2t-3t^2}}{2t^2(t+1)}.$$

Plugging this expression for $M_>(t,1,1)$ back into (8) we get that

$$M_>(t,1,v)=\frac{(1-t-2t^2v-\sqrt{1-2t-3t^2})v}{2t(1-v+tv+t^2v^2)}. \tag{9}$$

Again, we apply now the Kernel method to (7), taking $v=v_1=v_1(t,u)=\frac{1}{1-tu}$ and using (9), to get an expression for $M_>(t,u,1)$. Substituting back into (7) and using (9) again we get that

$$M_>(t,u,v)=\frac{[2-u-uv+u^2v+tu(v-1)-t(1+2t)u^2v+(1-2u)\sqrt{1-2t-3t^2})]tu^2v}{2(1-u-tu(1-u)+t^2u^2)(1-uv+tuv+t^2u^2v^2)}.$$

Finally, combining it with the fact that $M(t,u,v)=M_>(t,u,v)+M_=(t,u,v)=M_>(t,u,v)+tuv\,M_>(t,1,uv)$, we obtain the desired expression for $M(t,u,v)$. □

3.2. **{2-1-3, 32-1}-avoiding permutations.** It is known [3] that $|\mathcal{S}_n(\text{2-1-3},\text{32-1})|=2^{n-1}$. Here we use generating trees with two labels to recover this result, and to refine it with two parameters: the value of the last entry and the largest value of the bottom of a descent. Let $\mathcal{T}_2$ be the generating tree for the set of {2-1-3, 12-3}-avoiding permutations, where new elements are appended to the right of the permutation. Given any $\pi\in\mathcal{S}_n$, define the parameter

$$h(\pi)=\begin{cases}0 & \text{if } \pi=12\cdots n,\\ \max\{\pi_i : i>1,\ \pi_{i-1}>\pi_i\} & \text{otherwise.}\end{cases} \tag{10}$$

To each node π of $\mathcal{T}_2$ we assign the label $(h,r)=(h(\pi),r(\pi))$. Note that if π avoids {2-1-3, 32-1}, then necessarily $h\le r$. The proof of the following proposition and of similar ones below are straightforward and analogous to that of Proposition 3.1, so the details are left to the reader.

Proposition 3.3. *The generating tree $\mathcal{T}_2$ for {2-1-3, 32-1}-avoiding permutations is specified by the following succession rule on the labels:*

$$\begin{array}{l}(0,1)\\ (h,r)\longrightarrow(h+1,h+1)\ (h+1,h+2)\ \cdots\ (r,r)\ (h,r+1).\end{array}$$

We will use this description of the generating rule to obtain a formula for the generating function

$$N(t,u,v):=\sum_{n\ge1}\ \sum_{\pi\in\mathcal{S}_n(\text{2-1-3},\text{32-1})}u^{h(\pi)}v^{r(\pi)}\,t^n.$$

For fixed h and r, let $N_{h,r}(t)=\sum_{n\ge1}|\{\pi\in\mathcal{S}_n(\text{2-1-3},\text{32-1}):h(\pi)=h,r(\pi)=r\}|\,t^n$. Then we have that $N(t,u,v)=\sum_{h,r}N_{h,r}(t)u^hv^r$.

Theorem 3.4. *The generating function $N(t,u,v)$ enumerating {2-1-3, 32-1}-avoiding permutations with respect to size and the parameters h and r defined above can be expressed as*

$$N(t,u,v)=\frac{tv(1-t+tu-tuv)}{(1-tv)(1-t-tuv)}.$$

Proof. By Proposition 3.3, the children of a node with label (h,r) contribute $u^{h+1}v^{h+1}+u^{h+1}v^{h+2}+\cdots+u^rv^r+u^hv^{r+1}$ to the next level. It follows that

$$N(t,u,v)=tv+\sum_{h,r}N_{h,r}(t)\left[\frac{(uv)^{r+1}-(uv)^{h+1}}{uv-1}+u^hv^{r+1}\right]=tv+tvN(t,u,v)+\frac{tuv[N(t,1,uv)-N(t,uv,1)]}{uv-1}. \tag{11}$$

Substituting $u=1$ we get an equation relating $N(t,1,v)$ and $N(t,v,1)$, and similarly substituting $v=1$. Combining these two equations we get

$$N(t,1,v)=\frac{tv}{1-t-tv},\qquad N(t,u,1)=\frac{t}{1-t-tu}.$$

Using these expressions in (11) we get the desired formula for $N(t,u,v)$. □

3.3. **1-23-avoiding permutations.** It was shown in [3] that the number of 1-23-avoiding permutations of size n is the n-th Bell number. It is easy to obtain a generating tree with two labels for the class of 1-23-avoiding permutations if we take one of the labels to be the length n of the permutation. The label of $\pi\in\mathcal{S}_n$ is then (r,n), where $r=\pi_n$ as usual.

Proposition 3.5. *The generating tree for* 1-23*-avoiding permutations is specified by the following succession rule on the labels:*

$$\begin{array}{l}(1,1)\\ (r,n)\longrightarrow\begin{cases}(1,n+1)\ (2,n+1)\ \cdots\ (n+1,n+1) & \text{if } r=1,\\ (1,n+1)\ (2,n+1)\ \cdots\ (r,n+1) & \text{if } r>1.\end{cases}\end{array}$$

We define $G(t,u):=\sum_{n\geq1}\sum_{\pi\in\mathcal{S}_n(1\text{-}23)}u^{r(\pi)}\,t^n$. For fixed r, let $g_{n,r}=|\{\pi\in\mathcal{S}_n(1\text{-}23):r(\pi)=r\}|$, and let $G_r(t)=\sum_{n\geq1}g_{n,r}t^n$. Then we have that $G(t,u)=\sum_r G_r(t)u^r=\sum_{n,r}g_{n,r}u^rt^n$.

Theorem 3.6. *The generating function $G(t,u)$ enumerating* 1-23*-avoiding permutations with respect to size and the value of the rightmost entry is*

$$G(t,u)=\frac{1}{1-\frac{tu}{u-1}}\left[tu+t^2u^2+\frac{tu}{u-1}\sum_{k\geq1}\left(\frac{t^{k+1}u^{k+2}}{(1-tu)(1-2tu)\cdots(1-ktu)}-\frac{(1+tu)t^k}{(1-t)(1-2t)\cdots(1-kt)}\right)\right].$$

Proof. Proposition 3.5 gives the following recurrence for G.

$$\begin{aligned}G(t,u)&=tu+t\left[\sum_{r>1}G_r(t)\frac{u^{r+1}-u}{u-1}+\sum_{n\geq1}g_{n,1}t^n\frac{u^{n+2}-u}{u-1}\right]\\ &=tu+\frac{tu}{u-1}\left[(G(t,u)-uG_1(t))-(G(t,1)-G_1(t))+uG_1(tu)-G_1(t)\right]\end{aligned}\tag{12}$$

Since every node has exactly one child with $r=1$, we have that $G_1(t)=t+t\,G(t,1)$. Using this in (12) and collecting the terms with $G(t,u)$, we get

$$\left(1-\frac{tu}{u-1}\right)G(t,u)=tu+t^2u^2+\frac{tu}{u-1}\left[tu^2G(tu,1)-(1+tu)G(t,1)\right].\tag{13}$$

Applying the Kernel method with $u=\frac{1}{1-t}$ yields the functional equation $G(t,1)=\frac{t}{1-t}\left(1+G(\frac{t}{1-t},1)\right)$. By iterated application of this formula,

$$G(t,1)=\frac{t}{1-t}\left(1+\frac{t}{1-2t}\left(1+\frac{t}{1-3t}\left(1+\frac{t}{1-4t}\left(1+\cdots\right)\right)\right)\right)=\sum_{k\geq1}\frac{t^k}{(1-t)(1-2t)\cdots(1-kt)},$$

which is the well-known ordinary generating function for the Bell numbers. Substituting this expression back into (13) we get the desired formula for $G(t,u)$. □

3.4. 123**-avoiding permutations.** Permutations avoiding the consecutive pattern 123 were studied in [5], where the authors give their exponential generating function. We can describe a generating tree with two labels for this class of permutations. To each $\pi\in\mathcal{S}_n(123)$ we assign the label (π_n,n) if $\pi_{n-1}>\pi_n$ or $n=1$, and the label $(\pi_n,n)'$ if $\pi_{n-1}<\pi_n$.

Proposition 3.7. *The generating tree for* 123*-avoiding permutations is specified by the following succession rule on the labels:*

$$\begin{array}{lll}(1,1)&&\\ (r,n)&\longrightarrow&(1,n+1)\ (2,n+1)\cdots(r,n+1)\ (r+1,n+1)'\ (r+2,n+1)'\cdots(n+1,n+1)'\\ (r,n)'&\longrightarrow&(1,n+1)\ (2,n+1)\cdots(r,n+1).\end{array}$$

Let
$$A(t,u) := \sum_{n\geq 1} \sum_{\substack{\pi\in\mathcal{S}_n(123)\\ \pi_{n-1}>\pi_n\\ \text{or } n=1}} u^{r(\pi)}\ t^n, \qquad B(t,u) := \sum_{n\geq 1} \sum_{\substack{\pi\in\mathcal{S}_n(123)\\ \pi_{n-1}<\pi_n}} u^{r(\pi)}\ t^n,$$
and let $C(t,u) = A(t,u) + B(t,u)$. Proposition 3.7 provides the following functional equations defining A and B.
$$A(t,u) = tu + \frac{tu}{u-1}[C(t,u) - C(t,1)], \qquad B(t,u) = \frac{tu}{u-1}[uA(tu,1) - A(t,u)].$$
These equations have been solved by Mireille Bousquet-Mélou [2] using the Kernel method, obtaining the following expression for $C(t,1)$:
$$C(t,1) = \frac{3+i\sqrt{3}}{2(3t-i\sqrt{3})}\ C\left(\frac{t}{1+i\sqrt{3}t}, 1\right) - \frac{3(2t+1-i\sqrt{3})t}{(2t-1-i\sqrt{3})(3t-i\sqrt{3})}.$$
From this, one can obtain a recurrence for the coefficients of $C(t,1)$, and derive their exponential generating function
$$C(t,1) = \frac{\sqrt{3}}{2}\frac{e^{t/2}}{\cos(\frac{\sqrt{3}}{2}t + \frac{\pi}{6})}.$$

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