Some order-theoretic properties of the Motzkin and Schröder families*

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EXTENDED ABSTRACT

1 Dyck paths

In the present work we will consider three different kinds of combinatorial objects, namely paths, set partitions and permutations, trying to relate them by means of some natural partial order structure.

Our starting point is one of the main results of [BBFP], which we are going to recall in the next lines. Denote by $\mathcal{D}_n$, $NC(n)$ and $S_n(312)$ the sets of Dyck paths of length $2n$, noncrossing partitions of $[1,n]$ and 312-avoiding permutations of $[1,n]$, respectively, where $[1,n]$ is the set of positive integers less than or equal to n. For our purposes, the following notations will be particularly useful:

- Dyck paths will be represented either as finite words on the two letter alphabet $\{u,d\}$ or as functions with nonnegative integer values; in this latter interpretation, it is $\mathcal{D}_n = \{P : [0,2n] \to \mathbf{N} \mid P(0) = P(2n) = 0, |P(k+1) - P(k)| = 1, \forall k < 2n\}$;
- each noncrossing partition $\pi \in NC(n)$ is represented as $\pi = B_1|B_2|\cdots|B_k$, where the element inside each block B_i are in decreasing order, whereas the blocks are listed with their maxima in increasing order (so that $\max B_i < \max B_{i+1}$, for each $i < k$).

There are well known bijections linking $\mathcal{D}_n$, $NC(n)$ and $S_n(312)$. More precisely:

- Fix a Dyck path and label its up steps by enumerating them from left to right (so that the k-th up step is labelled k). Next assign to each down step the same label of the up step it is matched with. Now consider the partition whose blocks are constituted by the labels of each sequence of consecutive down steps. Such a partition is easily seen to be noncrossing. This correspondence, recalled for instance in [Si], is a bijection between $\mathcal{D}_n$ and $NC(n)$.

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- Given a noncrossing partition $\pi = B_1|\cdots|B_k$, removing the bars generates a 312-avoiding permutation. This correspondence is a bijection between $NC(n)$ and $S_n(312)$.

The set $\mathcal{D}_n$ can be endowed with a very natural and remarkable order structure [FP]: if $P, Q \in \mathcal{D}_n$, we define $P \leq Q$ when $P(i) \leq Q(i)$, for every $i \leq 2n$. The properties of the poset $[\mathcal{D}_n; \leq]$ have not been fully investigated yet; it can be shown that it is in fact a distributive lattice, whose rank function is essentially given by the area determined by a Dyck path and the x-axis. Such a distributive lattice will be called the *Dyck lattice* of order n.

For any $\pi = B_1|\cdots|B_k \in NC(n)$, denote by $\max(\pi)$ the vector whose i-th component is the maximum of the first i elements of π in the above canonical representation. We define the *Bruhat order* on $NC(n)$ by saying that $\pi \leq \rho$ when $\max(\pi) \leq \max(\rho)$. The resulting poset turns out to be a distributive lattice, which we call the *Bruhat noncrossing partition lattice* of order n. The following theorem contains some of the main results of [BBFP].

Theorem 1.1 *For any $n \in \mathbf{N}$, the following order structures are isomorphic:*

1. *the Dyck lattice $\mathcal{D}_n$;*
2. *the Bruhat noncrossing partition lattice $NC(n)$;*
3. *$S_n(312)$ as a subposet of S_n endowed with the strong Bruhat order.*

In particular, $S_n(312)$ is a distributive lattice with respect to the strong Bruhat order.

The above theorem can be proved essentially by showing that the above mentioned bijections are in fact order-isomorphisms.

Recalling [I] that the reverse and complement functions are antiisomorphisms and the inverse function is an isomorphism (with respect to the strong Bruhat order), an immediate consequence of theorem 1.1 is the following:

Proposition 1.1 *For every $n \in \mathbf{N}$, $S_n(312)$ is order-isomorphic to $S_n(231)$ and order-antiisomorphic to $S_n(132)$ and $S_n(213)$. Therefore all the above posets are distributive lattices. The posets $S_n(123)$ and $S_n(321)$ are not even lattices, since they do not have minimum and maximum, respectively.*

Clearly the posets $S_n(123)$ and $S_n(321)$ are antiisomorphic.

Open problem 1. Describe the poset $S_n(123)$.

Open problem 2. Fixed $k \in \mathbf{N}$, $k > 3$, for which $\tau \in S_k$ is $S_n(\tau)$ a (distributive) lattice? In case of a positive answer, is it possible to give some alternative combinatorial descriptions of such lattices?

2 Motzkin paths

In this section we try to develop the above considerations in the case of Motzkin paths. Also in this case, introducing a partial order analogous to that of Dyck paths, the set $\mathcal{M}_n$ of Motzkin paths of length n is a distributive lattice

[FP], called the *Motzkin lattice* of order n. Here the rank of a Motzkin path coincides with the area determined by the path itself and the x-axis.

We start by recalling a bijection considered by Elizalde and Mansour [EM] between $\mathcal{M}_n$ and the set $\mathcal{D}_n^{(3)}$ of Dyck paths of length $2n$ without three consecutive down steps. Every Dyck path $P \in \mathcal{D}_n^{(3)}$ can be uniquely decomposed into factors of the following three types: u, ud, udd. Define a Motzkin path $f(P)$ by translating the above factors according to the following:

$$\begin{array}{rcl} u & \to & u \\ ud & \to & h \\ udd & \to & d, \end{array}$$

where h denotes the horizontal step. The path $f(P)$ has length n and it is possible to show that the function f is a bijection. Our next proposition shows that f has some more structural properties.

Proposition 2.1 *The function $f : \mathcal{D}_n^{(3)} \longrightarrow \mathcal{M}_n$ is an order-isomorphism.*

The bijection between $\mathcal{D}_n$ and $NC(n)$ recalled in section 1 can be restricted to $\mathcal{D}_n^{(3)}$; the corresponding subset of $NC(n)$ is easily seen to consist of noncrossing partitions whose blocks have cardinality at most 2. Call such partitions *Motzkin noncrossing partitions.* Thanks to the last proposition we can establish the following result.

Theorem 2.1 *The set $MNC(n)$ of Motzkin noncrossing partitions of $[1, n]$ can be endowed with a distributive lattice structure, which is isomorphic to the lattice of Motzkin paths of length n. More precisely, given $\pi, \rho \in MCN(n)$, we have that $\pi \prec \rho$ (i.e., π is covered by ρ) if ρ is obtained from π by moving the minimum of some block B of π into the block $\tilde{B}$ containing the element $\beta = \max B + 1$ if $\beta = \min \tilde{B}$. In this case, either:*

1. *keep β inside $\tilde{B}$, if $|\tilde{B}| = 1$, or*
2. *add a new block $\widehat{B} = \{\beta\}$, if $|\tilde{B}| = 2$.*

Example. Given the partition $2|31|65|74|8 \in MNC(n)$, there are two partitions covering it, which are $2|3|4|65|71|8$ (1 is moved into a block with two elements) and $2|31|65|7|84$ (4 is moved into a block with one element). Note that we cannot move neither 2 nor 5, since the elements 3 and 7 are not the minima of their blocks.

Similarly to [BBFP], it is possible to transfer the distributive lattice structure on Motzkin noncrossing partitions to a suitable class of pattern avoiding permutations, via a bar-removing bijection. In [C] it is shown that $S_n(3-21, 31-2)$ is counted by Motzkin numbers. Here we give a bijection between $MNC(n)$ and $S_n(3-21, 31-2)$.

Proposition 2.2 *Removing the bars in Motzkin noncrossing partitions defines a bijection between $MNC(n)$ and the set $S_n(3-21, 31-2)$ of pattern avoiding permutations of $[1, n]$, for any $n \in \mathbf{N}$.*

To prove that the above bar-removing bijection between $MNC(n)$ and $S_n(3-21, 31-2)$ is also an order-isomorphism, we just notice that such a bijection is obtained by simply restricting the bar-removing isomorphism between $NC(n)$ and $S_n(312)$ considered in [BBFP]. Therefore the following theorem holds.

Theorem 2.2 *Let $(S_n(3-21, 31-2); \leq)$ be the poset obtained by transferring the distributive lattice structure defined in theorem 2.1 along the bar-removing bijection. This is precisely the subposet induced on $S_n(3-21, 31-2)$ by the strong Bruhat order of the symmetric group S_n. Therefore $S_n(3-21, 31-2)$ is a distributive sublattice of S_n endowed with the strong Bruhat order.*

An immediate consequence of the preceding theorem is stated in the following, remarkable corollary.

Corollary 2.1 *For any $n \in \mathbf{N}$, the Motzkin lattice $\mathcal{M}_n$ is isomorphic to the lattice $S_n(3-21, 31-2)$ with the strong Bruhat order.*

3 Schröder paths

In this section we try to find analogous results starting from Schröder paths. The set $\mathcal{S}_n$ of Schröder paths of length $2n$ is a distributive lattice [FP]. Also in this case, for two Schröder paths of length $2n$, having the same area means having the same rank in $\mathcal{S}_n$.

The key idea consists of interpreting Schröder paths as Dyck paths with bicoloured peaks. Denote by $\overline{\mathcal{D}}_n$ the set of Dyck paths of length $2n$ whose peaks can possibly be coloured. There is an obvious bijection between $\overline{\mathcal{D}}_n$ and the set $\mathcal{S}_n$ of Schröder paths of length $2n$ (just map noncoloured peaks into simple peaks, coloured peaks into a pair of consecutive horizontal steps, and leave the remaining steps unchanged; from this bijection, which has been considered in [Su], immediately follows the identity $R_n = \sum_{k=1}^{n} 2^k N(n,k)$, where R_n denotes the n-th Schröder number). Thanks to this simple observation, it is not difficult to find a suitable set of coloured noncrossing partitions in bijection with Schröder paths.

Proposition 3.1 *Denote by $\overline{NC}(n)$ the set of noncrossing partitions of $[1,n]$ such that the maximum of each block can possibly be coloured. Then there is a bijection between $\mathcal{S}_n$ and $\overline{NC}(n)$.*

The order structure on $\mathcal{S}_n$ can be transferred on $\overline{NC}(n)$ by means of the above bijection. The resulting lattice will be called the *Schröder (type B) noncrossing partition lattice* of order n. Therefore we have the following theorem:

Theorem 3.1 *(Characterization of coverings in $\overline{NC}(n)$) Given two coloured noncrossing partitions $\pi, \rho \in \overline{NC}(n)$, we have $\pi \prec \rho$ if and only if ρ is obtained from π by either*

1. *removing the colour from an element of π, or*
2. *moving the minimum of some block B of π into the block $\tilde{B}$ containing the element $\beta = \max B + 1$ only when β is not coloured; moreover:*

(a) *if* $\beta = \max \tilde{B}$, *then keep* β *inside* $\tilde{B}$ *and colour it;*

(b) *if* $\beta \neq \max \tilde{B}$, *then add the block* $\widehat{B} = \{\overline{\beta}\}$.

Example. Given the partition $\bar{5}43|62|871|\bar{9} \in \overline{NC}(n)$, there are precisely four partitions covering it, which are $543|62|871|\bar{9}$ ($\bar{5}$ is not coloured), $\bar{5}4|\bar{6}32|871|\bar{9}$ (3 is moved and 6 is the maximum of its block), $\bar{5}43|6|\bar{7}|821|\bar{9}$ (2 is moved and 7 is not the maximum of its block) and $\bar{5}43|62|871|9$ ($\bar{9}$ is not coloured). Note that the partition obtained by moving 1 into the block containing 9 (i. e. the maximum of its block plus 1) is not listed above, since 9 is coloured.

Following the same lines of [BBFP], we now look for a suitable set of coloured pattern avoiding permutations in bijection with both Schröder paths and Schröder noncrossing partitions. In what follows, we denote by $\overline{S_n}$ the set of *coloured permutations* of $[1, n]$, i.e. permutations whose elements can possibly be coloured. The study of the enumerative properties of coloured pattern avoiding permutations has been pursued by several authors, see for example [M]. Our next result implies that a certain class of coloured pattern avoiding permutations is enumerated by Schröder numbers. This fact has been independently proved by Egge [E] using algebraic arguments; here we propose a bijective proof, as well as a presumably new order structure connecting such class of permutations with Schröder paths and Schröder noncrossing partitions.

Theorem 3.2 *Removing the bars in coloured noncrossing partitions defines a bijection between* $\overline{NC}(n)$ *and the set* $\overline{S}_n(312, \bar{2}\bar{1}, 2\bar{1}, \bar{3}12)$, *for any* $n \in \mathbf{N}$.

Using the above bar-removing bijection we can now transfer the order structure of Schröder paths on the set $\overline{S}_n(312, \bar{2}\bar{1}, 2\bar{1}, \bar{3}12)$. What we obtain is clearly a distributive lattice, whose covering relation is described in the next proposition.

Proposition 3.2 *Given* $\pi, \rho \in \overline{S}_n(312, \bar{2}\bar{1}, 2\bar{1}, \bar{3}12)$, *it is* $\pi \prec \rho$ *if and only if* ρ *is obtained from* π *by either:*

1. *removing the colour from an element of* π, *or*

2. *interchanging the last element* a *of a descent of* π *with* β, *where* $\beta - 1$ *is the first element of that descent, and colouring* β*; this last operation can be performed exclusively when* a *and* β *are both uncoloured.*

Remark. We recall that it is possible to define a notion of Bruhat order on coloured permutations, as it is reported, for instance, in [BB]. Unfortunately, the restriction of this Bruhat order to $\overline{S}_n(312, \bar{2}\bar{1}, 2\bar{1}, \bar{3}12)$ does not match our poset.

Open problem 3. Concerning the above remark, the Bruhat order on $\overline{S}_n$ is defined as the Bruhat order on the set of permutations with ground set $\{1, \ldots n, \overline{1}, \ldots \overline{n}\}$, where the elements are linearly ordered as they are listed above (i.e., $1 < \cdots < n < \overline{1} < \cdots < \overline{n}$). Is it possible to fine a suitable linear order on $\{1, \ldots n, \overline{1}, \ldots \overline{n}\}$ such that the resulting Bruhat order on $\overline{S}_n$ coincides with our partial order?

Let $\pi \in \overline{S}_n$; we denote by $inv(\pi)$ the set of the inversions of π and $nb(\pi)$ the number of the uncoloured entries of π. Then the following proposition holds.

Proposition 3.3 *The rank of an element* π *in the lattice* $\overline{S}_n(312, \bar{2}\bar{1}, 2\bar{1}, \bar{3}12)$ *is given by*

$$r(\pi) = 2|inv(\pi)| + nb(\pi) \ .$$

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