The first appearance of a 123 pattern

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Let $\pi \in S^+_{\mathbf{Z}}$ denote a random one-to-one function from $\mathbf{Z}^+ = \{1, 2, \ldots\}$ to $\mathbf{Z}^+$. Order the 3-subsets (written in an increasing ordered fashion) of $\mathbf{Z}^+$ lexicographically as

$$\{1, 2, 3\}, \{1, 2, 4\}, \ldots \{1, 3, 4\}, \{1, 3, 5\} \ldots \{2, 3, 4\} \ldots$$

It is then clear that the first 1-2-3 pattern X must occur at positions $\{1, s, t\}$ for some s, t since given $\pi(1)$ there must exist an $s \geq 2$ with $\pi(s) > \pi(1)$ and a $t > s$ such that $\pi(t) > \pi(s)$. We find, for $2 \leq s < t < \infty$ the probability

$$p_{1,s,t} = \mathbb{P}(\text{the first } 1-2-3 \text{ pattern occurs at positions } \{1, s, t\}).$$

The situation gets far more complicated when $\pi \in S_n$ is a random permutation of the first n integers since then the first 1-2-3 pattern can occur at any 3-subset of $\{1, 2, \ldots, n\}$. We provide an explicit formula for $p_{1,s,t}; 2 \leq s < t \leq n$ and prove several facts regarding $p_{r,s,t} : 2 \leq r < s < t \leq n$. Among these are

$$\mathbb{P}(\text{the first } 1-2-3 \text{ pattern occurs at or after position } \{2, 3, 4\}) \sim \frac{e}{n}.$$

Various summary statistics for X such as means and quantiles are derived both in the finite and infinite cases.