DEODHAR ELEMENTS AND EMBEDDED FACTOR PATTERNS

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Abstract. Deodhar [Deo90] proposes a combinatorial framework for determining the Kazhdan-Lusztig polynomials of an arbitrary Coxeter group. The algorithm he describes is shown to work for all Weyl groups, but typically involves recursion. However, under a certain condition on the element w of a Weyl group, Deodhar's algorithm for determining the Kazhdan-Lusztig polynomial $P_{x,w}$ associated to the elements x and w, for any $x \leq w$ in the Weyl group, turns out to be a simple combinatorial formula. We say that w is *Deodhar* when it satisfies the condition. In this paper we characterize the Deodhar property for elements of finite Weyl groups using a new form of pattern avoidance inspired by reduced expressions and heaps, rather than the 1-line notations on which the traditional forms of pattern avoidance rely. The embedded factor pattern avoidance turns out to be more efficient than the standard pattern avoidance (although either can be used to detect the Deodhar condition). In order to perform this characterization, we enhance the definitions and techniques found in [BW01], which showed that the Deodhar condition for type A is equivalent to avoiding short-braids, and a single additional embedded factor pattern called a hexagon.

1. Extended abstract

The Kazhdan-Lusztig polynomials for finite Weyl groups arise as Poincaré polynomials for intersection cohomology of Schubert varieties [KL80] and as a q-analog of the multiplicities for Verma modules [BB81, BK81]. They are defined to be the coefficients in the transition matrix for expanding the Kazhdan-Lusztig basis elements in the Hecke algebra associated to the Weyl group into the standard basis. Several recursive algorithms exist, formulas for special cases, and interesting properties are known for these polynomials; see for example [MW02, Pol99, LS81, Bre04, Deo94, Hum90]. In particular, these polynomials have nonnegative integer coefficients but no explicit combinatorial interpretation for the coefficients is known in general.

Deodhar [Deo90] proposes a combinatorial framework for determining the Kazhdan-Lusztig polynomials of an arbitrary Coxeter group using a combinatorial approach. The algorithm he describes is shown to work for all Weyl groups where the Kazhdan-Lusztig polynomials are known to have nonnegative integer coefficients, which includes affine and finite Weyl groups, but typically involves recursion. However, under certain conditions, Deodhar's algorithm for determining some of the Kazhdan-Lusztig polynomials turns out to be a beautiful combinatorial formula. We say that a Weyl group element w is *Deodhar* when it satisfies these conditions.

In 1999, Billey and Warrington [BW01] gave an efficient characterization of the Deodhar elements in the symmetric group as *321-hexagon avoiding* permutations. Their results extend to finite *linear* Weyl groups, types A, B, F, G. Our goal is to give a similar characterization for all finite Weyl groups. Traditional pattern avoidance and the generalized pattern avoidance for Coxeter groups in [BP05, BB02] lead to long lists of patterns in types D and E necessary to characterize the Deodhar elements. We describe a new type of pattern that we call *embedded*

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factors which can be defined for all Coxeter groups and lead to a short list of minimal patterns for the Deodhar elements in finite Weyl groups.

Our main theorem states that the Deodhar elements of Weyl groups can be characterized by avoiding embedded factors from the following list, as well as an additional 1-line pattern for type D. This characterization can be translated into a polynomial time algorithm by using generalized pattern avoidance with a finite but long list of patterns. For example, the Weyl group of type E_8 has parabolic subgroups of types A_2, A_7, D_6, E_6, and E_7 from this list. Therefore, a Deodhar element w in the Weyl group of type E_8 cannot be written in the form $w = avb$ where $l(w) = l(a)+l(v)+l(b)$ and v has a reduced expression of the form of a short-braid, the hexagon, HEX_5, or any of the E_6 or E_7 patterns on the corresponding parabolic subgroup. In type D_n, the Deodhar elements must also avoid the "D-hexagon" $[-1, 6, 7, 8, -5, 2, 3, 4]$ as a generalized 1-line pattern.

Lie Type	Coxeter Graph	Embedded Factor Patterns
A_2	$\bullet_1$ — $\bullet_2$	$s_1s_2s_1$, $s_2s_1s_2$ (short-braids)
A_7	$\bullet_1$ — $\bullet_2$ — $\bullet_3$ — $\bullet_4$ — $\bullet_5$ — $\bullet_6$ — $\bullet_7$	$s_5s_6s_7s_3s_4s_5s_6s_2s_3s_4s_5s_1s_2s_3$ (hexagon)
B_2	$\bullet_0 \overset{4}{—} \bullet_1$	$s_0s_1s_0$ (short-braid)
B_7/C_7	$\bullet_0 \overset{4}{—} \bullet_1$ — $\bullet_2$ — $\bullet_3$ — $\bullet_4$ — $\bullet_5$ — $\bullet_6$	$s_4s_5s_6s_2s_3s_4s_5s_1s_2s_3s_4s_0s_1s_2$ (hexagon)
$G_2 = I_2(6)$	$\bullet_1 \overset{6}{—} \bullet_2$	$s_1s_2s_1$ (short-braid)
D_6	$\bullet_{\tilde{1}}$ \\ $\bullet_1$ — $\bullet_2$ — $\bullet_3$ — $\bullet_4$ — $\bullet_5$	$s_3s_4s_5s_1s_2s_3s_4s_{\tilde{1}}s_2s_3s_1$ (HEX_5)
D_7	$\bullet_{\tilde{1}}$ \\ $\bullet_1$ — $\bullet_2$ — $\bullet_3$ — $\bullet_4$ — $\bullet_5$ — $\bullet_6$	$s_3s_4s_5s_6s_2s_3s_4s_5s_{\tilde{1}}s_2s_3s_4s_1s_2s_3$ (HEX_2) $s_4s_5s_6s_2s_3s_4s_5s_1s_2s_3s_4s_{\tilde{1}}s_2s_1$ (HEX_{3a}) $s_1s_4s_5s_6s_2s_3s_4s_5s_{\tilde{1}}s_2s_3s_4s_1s_2$ (HEX_{3b})
D_8	$\bullet_{\tilde{1}}$ \\ $\bullet_1$ — $\bullet_2$ — $\bullet_3$ — $\bullet_4$ — $\bullet_5$ — $\bullet_6$ — $\bullet_7$	$s_4s_5s_6s_7s_3s_4s_5s_6s_2s_3s_4s_5s_{\tilde{1}}s_1s_2s_3s_4$ (D-hexagon, to be avoided as a 1-line pattern)
E_6	$\bullet_5$ (joined to $\bullet_2$) $\bullet_0$ — $\bullet_1$ — $\bullet_2$ — $\bullet_3$ — $\bullet_4$	$s_0s_1s_2s_5s_3s_4s_2s_3s_1s_2s_5s_0s_1$ $s_5s_1s_2s_3s_0s_1s_2s_5s_4s_3s_2s_1s_0$ $s_1s_2s_5s_3s_4s_2s_3s_1s_2s_5s_0s_1s_2$ $s_2s_5s_1s_2s_3s_0s_1s_2s_5s_4s_3s_2s_1$
E_7	$\bullet_5$ (joined to $\bullet_2$) $\bullet_0$ — $\bullet_1$ — $\bullet_2$ — $\bullet_3$ — $\bullet_4$ — $\bullet_5$	$s_0s_1s_2s_3s_4s_6s_5s_2s_3s_4s_1s_2s_3s_0s_1$ $s_3s_4s_6s_1s_2s_3s_0s_1s_2s_5s_4s_3s_2s_1s_0$ $s_1s_2s_3s_4s_6s_5s_2s_3s_4s_1s_2s_3s_0s_1s_2$ $s_2s_3s_4s_6s_1s_2s_3s_0s_1s_2s_5s_4s_3s_2s_1$ $s_5s_2s_3s_4s_6s_1s_2s_5s_3s_4s_2s_3s_0s_1s_2s_5$

FIGURE 1. Minimal non-Deodhar patterns

Let W be a Coxeter group with generating set S and relations of the form $(s_i s_j)^{m(i,j)} = 1$. For a reader unfamiliar with Coxeter groups, we recommend either the classic text by Humphreys [Hum90] or the recent text by Björner and Brenti [BB05]. The Coxeter graph for W is the graph on the generating set S with edges connecting s_i and s_j labeled $m(i,j)$ for all pairs i, j with $m(i,j) > 2$. For example, the table in Figure 1 shows the Coxeter graphs for the finite Weyl groups which contain minimal non-Deodhar patterns. Note, if $m(i,j) = 3$ it is customary to leave the corresponding edge unlabeled.

An *expression* is any product of generators from S. The *length* $l(w)$ of an element $w \in W$, is the minimum length of any expression for the element w. Such a minimum length expression is called *reduced*. Each element $w \in W$ can have several different reduced expressions which represent it. Given $w \in W$, we represent a reduced expressions for w in sans serif font, say $\mathsf{w} = \mathsf{w}_1 \mathsf{w}_2 \cdots \mathsf{w}_p$ where each $\mathsf{w}_i \in S$. When W is the symmetric group, we may represent permutations uniquely with *one-line notation* by $w = [w_1 \dots w_n]$ where w is the bijection mapping i to w_i.

Let $w, y \in W$. We say that w contains y *as a factor* if there exist elements a and b in W such that $w = ayb$ and $l(w) = l(a) + l(y) + l(b)$. Equivalently, w contains y as a factor if some reduced expression for w contains some reduced expression for y as a consecutive subword, i.e. $\mathsf{w} = \mathsf{w}_1 \mathsf{w}_2 ... \mathsf{w}_p$ and $y = \mathsf{w}_i \mathsf{w}_{i+1} \cdots \mathsf{w}_j$ for some $1 \leq i \leq j \leq p$. The induced partial order on Coxeter group elements is known as the *two-sided weak Bruhat order* [BB05].

Our first result is:

Theorem 1.1. *If y is not Deodhar and w contains y as a factor, then w is not Deodhar either.*

This theorem implies that the non-Deodhar elements form an upper order ideal in the two-sided weak Bruhat order. In order to obtain a generating set for this ideal, we consider a refinement of factor containment.

Definition 1.2. Let W be a Coxeter group with associated Coxeter graph G. For any $p \in W$, let G_p be the induced subgraph of G with verticies in the support of p, and let W_p be the parabolic subgroup generated by the support of p. Then, a *Coxeter embedding of* G_p is an injective map of the generators $f : G_p \to G$ which restricts to a labeled graph isomorphism onto its image.

A Coxeter embedding induces an injection from W_p to W, and we will abuse notation and call this map $f : W_p \to W$ a Coxeter embedding also.

It follows from the definition of the Deodhar condition that p is Deodhar if and only if $f(p)$ is Deodhar, for any Coxeter embedding $f : G_p \to G$. This leads to the central definition for our characterization result.

Definition 1.3. Suppose W is a Coxeter group, and $y \in W$. Let Y be the parabolic subgroup whose generators are determined by the support of y. If $f : Y \to W$ is a Coxeter embedding, and $w \in W$ contains y as a factor, then we say that w contains $f(y)$ *as an embedded factor*, denoted $y \preceq w$.

This definition yields a stronger reformulation of Theorem 1.1, which enables us to use shorter lists of non-Deodhar patterns.

Corollary 1.4. *Let y be a Coxeter element that is not Deodhar. If w is a Coxeter element that contains y as an embedded factor, then w is not Deodhar either.*

Here are a few examples of embedded factors.

Example 1.5. In type A, the Coxeter embeddings of connected subgraphs are simply shifts of the generators along the linear Coxeter graph or shifts of the dual element. In particular, if the generators are labelled so that $s_1, s_2, \dots, s_l$ form a path in the Coxeter graph, then the images of the reduced expression $s_{i_1}s_{i_2}\dots s_{i_k}$ are all of the form $s_{i_1+j}s_{i_2+j}\dots s_{i_k+j}$. Additionally, there is one other Coxeter embedding which realizes the dual of an element, mapping s_i to s_{n-i+1} for each $i \in \{1, \dots, n\}$.

Example 1.6. In the Coxeter group of type B_4, $[2\bar{4}51\bar{3}] = s_1s_2s_3s_1s_0s_1s_2s_1s_0s_4s_3s_4s_1$ contains the factor $s_4s_3s_4$ in the parabolic subgroup generated by the support s_3, s_4. This subgroup is isomorphic to S_3 and $s_4s_3s_4$ maps to $s_2s_1s_2 = [321] \in S_3$, so $[2\bar{4}51\bar{3}]$ contains $[321]$ as an embedded factor.

There are several relevant partial orders on Coxeter groups that we use for pattern containment:

Notation	Partial order
$p \leq_2 w$	Two-sided weak Bruhat order
$p \preceq w$	Embedded factor containment
$p \trianglelefteq w$	Classical permutation pattern containment

We compare embedded factor pattern containment in type A with classical permutation pattern avoidance.

Definition 1.7. We say that p is *connected* if $\text{supp}(p)$ forms a connected subset of the Coxeter graph.

Definition 1.8. Let $u = [u_1u_2\dots u_n] \in S_n$, and let $i_1, i_2, \dots, i_k$ be marked positions in the 1-line notation for u. Then, we say $v \in S_k$ is a *flattening of u with respect to positions $i_1, i_2, \dots, i_k$* if v is the unique element of S_k for which u contains v as a permutation pattern in the marked positions. That is, the 1-line notation for v is order-isomorphic to $[u_{i_1}u_{i_2}\dots u_{i_k}]$.

Theorem 1.9. *Suppose $w \in S_n$ and $p \in S_k$ is connected. If p is an embedded factor of w, then there exists $q \in S_k$ and a Coxeter embedding $g : S_k \to S_k$ such that $q \geq_2 g(p)$ and w contains q as a permutation pattern.*

In order to study when the converse to this theorem holds, we make the following definition.

Definition 1.10. An element p of S_k is called an *ideal pattern* if avoiding it as an embedded factor is equivalent to avoiding the upper order ideal $U_p = \bigcup_g \{q \in S_k : q \geq_2 g(p)\}$ as 1-line permutation patterns, where $g : S_k \to S_k$ ranges over all Coxeter embeddings.

Proposition 1.11. *Let $p \in S_k$. Then, p is an ideal pattern if and only if whenever w contains p as a permutation pattern, it also contains p as an embedded factor pattern.*

Tenner [Ten05] defines a notion of factor containment by considering linear shifts of reduced expressions rather than equivalence by arbitrary Coxeter embeddings. However, if the permutation is connected and we ignore the dual Coxeter embedding, this amounts to the same thing. Hence, this work shows that on a single connected component, vexillary patterns (i.e. those whose 1-line avoids the permutation pattern 2143) satisfy the converse of 1.9.

Theorem 1.12. [Ten05] *Let $p \in S_k$ be connected. If p is vexillary, then p is an ideal pattern.*

It is possible to refine this result slightly using the more general definition of embedded factor pattern containment.

Definition 1.13. Recall that p is *connected* if $\text{supp}(p)$ forms a connected subset of the Coxeter graph. If $\text{supp}(p)$ consists of multiple connected components $U_1, \dots, U_m$ in the Coxeter graph, then we can write p as a product $u_1u_2\dots u_m$ where each u_i is an element of the parabolic subgroup generated by U_i. We denote this situation by $p = u_1 \oplus \cdots \oplus u_m$.

Proposition 1.14. *Let $p \in A_k$. If $p = u_1 \oplus u_2 \cdots \oplus u_m$, then p is an ideal pattern if and only if all of the u_i are ideal patterns.*

Example 1.15. Consider that $w = [\mathbf{21354}] = s_1 s_4$ contains $p = [2143] = s_1 s_3$, both as a 1-line permutation pattern, but also as an embedded factor pattern because we have the Coxeter embedding f which maps $s_1 \mapsto s_1$ and $s_3 \mapsto s_4$. Observe that p has two connected components, each of which are vexillary, and p is ideal.

On the other hand, $w = [\mathbf{251364}] = s_4 s_1 s_3 s_5 s_2$ contains $p = [24153] = s_1 s_3 s_2 s_4$ as a permutation pattern, but *not* as an embedded factor. This demonstrates that p is not ideal. Note that p has a single connected component, and it is not vexillary.

We also obtain some preliminary enumerative results for embedded factor containment. In particular, we obtain a Stanley-Wilf bound for embedded factor pattern classes via [MT04] by relating the enumeration of embedded factor classes to that of classical permutation pattern classes.

There has been some work on the enumeration of 1-line patterns in other Coxeter types (e.g. [MW04] enumerates all classes in B_n with a basis from B_2), but the enumeration of embedded factor pattern classes is a completely unexplored area.

In contemplating the precise enumeration of ideal embedded factor pattern classes, it would be interesting to investigate whether the ideal condition (that the class be closed in the $\leq_2$ order) makes the enumeration any easier than for arbitrary permutation pattern classes.

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