Permutation Patterns 2006
Reykjavik University, Reykjavik, Iceland

Classifying Descents According to Equivalence mod k

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Abstract. In [**5**] the authors refine the well-known permutation statistic "descent" by fixing parity of one of the descent's numbers. In this paper, we generalize the results of [**5**] by studying descents according to whether the first or the second element in a descent pair is equivalent to k mod $k \geq 2$. We provide either an explicit or an inclusion-exclusion type formula for the distribution of the new statistics. Based on our results we obtain combinatorial proofs of a number of remarkable identities. We also provide bijective proofs of some of our results.

1. Introduction

The *descent set*, $Des(\pi)$, of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is the set of indices i for which $\pi_i > \pi_{i+1}$. The number of *descents* in a permutation π, denoted by $des(\pi)$, is a classical permutation statistic. This statistic was first studied by MacMahon [**8**] almost a hundred years ago, and it still plays an important role in the study of permutation statistics.

In [**5**], the authors considered counting descents according to the parity of the first or second element of the descent pair. In this paper, we generalize Kitaev and Remmel's results [**5**] by studying the problem of counting descents according to whether the first or the second element in a descent pair is equivalent to 0 mod k for $k \geq 2$. For any $k > 0$, let $kN = \{0, k, 2k, 3k, \ldots\}$. Given $X \subseteq N = \{0, 1, \ldots\}$ and $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n$, we define the following:

- $\overleftarrow{Des}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1} \ \&\ \sigma_i \in X\}$ and $\overleftarrow{des}_X(\sigma) = |\overleftarrow{Des}_X(\sigma)|$;
- $\overrightarrow{Des}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1} \ \&\ \sigma_{i+1} \in X\}$ and $\overrightarrow{des}_X(\sigma) = |\overrightarrow{Des}_X(\sigma)|$;
- $A_n^{(k)}(x) = \sum_{\sigma\in\mathcal{S}_n} x^{\overleftarrow{des}_{kN}(\sigma)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} A_{j,n}^{(k)} x^j$.
- $B_n^{(k)}(x) = \sum_{\sigma\in\mathcal{S}_n} x^{\overrightarrow{des}_{kN}(\sigma)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} B_{j,n}^{(k)} x^j$.
- $B_n^{(k)}(x,z) = \sum_{\sigma\in\mathcal{S}_n} x^{\overrightarrow{des}_{kN}(\sigma)} z^{\chi(\sigma_1\in kN)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \sum_{i=0}^{1} B_{i,j,n}^{(k)} z^i x^j$.

Remark 1. *Note that setting $k = 1$ gives us (usual) descents providing $A_n^{(1)}(x) = B_n^{(1)}(x) = A_n(x)$, whereas setting $k = 2$ gives $\overleftarrow{Des}_E(\sigma)$ and $\overrightarrow{Des}_E(\sigma)$ studied in* [**5**].

Kitaev and Remmel [**5**] showed that there are some surprisingly simple formulas for the coefficients of these polynomials. For example, they proved

2000 *Mathematics Subject Classification.* Primary 05A15; Secondary 05E10.
Key words and phrases. pattern matching, permutations, descents.
The second author is supported in part by NSF grant DMS 0400507.

Theorem 1.

$$
\begin{aligned}
A^{(2)}_{k,2n} &= \binom{n}{k}^2 (n!)^2,\\
A^{(2)}_{k,2n+1} &= \frac{1}{k+1}\binom{n}{k}^2 ((n+1)!)^2,\\
B^{(2)}_{1,k,2n} &= \binom{n-1}{k}\binom{n}{k+1}(n!)^2,\\
B^{(2)}_{0,k,2n} &= \binom{n-1}{k}\binom{n}{k}(n!)^2,\\
B^{(2)}_{0,k,2n+1} &= (k+1)\binom{n}{k}\binom{n+1}{k+1}(n!)^2 = (n+1)\binom{n}{k}^2(n!)^2, \text{ and}\\
B^{(2)}_{0,k,2n+1} &= (n+1)\binom{n}{k}^2(n!)^2.
\end{aligned}
$$

The goal of this paper is to derive closed formulas for the coefficients of these polynomials. When $k > 2$, our formulas are considerably more complicated than the formulas in the $k = 2$ case. In fact, in most cases, we can derive two distinct formulas for the coefficients of these polynomials. We shall see that there are simple recursions for the coefficients of the polynomials $A^{(k)}_{kn+j}(x)$, $B^{(k)}_{kn+j}(x)$, and $B^{(k)}_{kn+j}(x,z)$ for $0 \le j \le k-1$. In fact, we can derive two different formulas for the coefficients of our polynomials by iterating the recursions starting with the constant term and by iterating the recursions starting with highest coefficient. For example, we shall prove the following theorem.

Theorem 2. *For all $k \ge 2$, $n \ge 0$, and $0 \le j \le k-1$,* $A^{(k)}_{s,kn+j} =$

$$
\begin{aligned}
&((k-1)n+j)!\sum_{r=0}^{s}(-1)^{s-r}\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{s-r}\prod_{i=0}^{n-1}(r+1+j+(k-1)i) =\\
&((k-1)n+j)!\sum_{r=0}^{n-s}(-1)^{n-s-r}\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{n-s-r}\prod_{i=1}^{n}(r+(k-1)i) \qquad (1)
\end{aligned}
$$

These two different formulas for $A^{(k)}_{kn+j}$ lead to a number of identities that are interesting in the own right. Even in the case $k = 2$, we get some remarkable identities: for all $n \ge s$,

$$
\begin{aligned}
\binom{n}{s}^2 (n!) &= \sum_{r=0}^{s}(-1)^{s-r}\binom{n+r}{r}\binom{2n+1}{s-r}\prod_{i=0}^{n-1}(r+1+i)\\
&= \sum_{r=0}^{n-s}(-1)^{n-s-r}\binom{n+r}{r}\binom{2n+1}{n-s-r}\prod_{i=1}^{n}(r+i) \qquad (2)
\end{aligned}
$$

It turns out that both of these identities can be derived by using certain hypergeometric series identities. For example, we will show how (2) can be derived from Saalcshütz's identity. Jim Haglund [**4**] suggested that (1) should follow from Gasper's transformation [**2**] of hypergeometric series of Karlsson-Minton type. This is indeed the case but we will not include such a derivation in this paper since (1) is a special case of wider class of identities that arise by studying the problem of enumerating permutations according to the number of pattern matches where the equivalence classes of the elements modulo k for $k \ge 2$ are taken into account, see [**6**]. A general derivation of this wider class of identities from the Gasper's transformation of hypergeometric series of Karlsson-Minton type will appear in a subsequent paper [**7**].

2. Properties of $A^{(k)}_n(x)$

For $j = 1,\dots,k-1$, let Δ_{kn+j} be the operator which sends x^s to $sx^{s-1} + (kn+j-s)x^s$ and Γ_{kn+k} be the operator that sends x^s to $(s+1)x^s + (kn+k-1-s)x^{s+1}$. Then we have the following.

Theorem 3. *The polynomials $\{A^{(k)}_n(x)\}_{n\ge1}$ satisfy the following recursions.*

(1) $A_1^{(k)}(x) = 1$,
(2) *For* $j = 1, \ldots, k-1$, $A_{kn+j}^{(k)}(x) = \Delta_{kn+j}(A_{kn+j-1}^{(k)}(x))$ *for* $n \geq 0$, *and*
(3) $A_{kn+k}^{(k)}(x) = \Gamma_{kn+k}(A_{kn+k-1}^{(k)}(x))$ *for* $n \geq 1$.

It is easy to see from Theorem 3 that we have two following recursions for the coefficients $A_{s,n}^{(k)}$.
For $1 \leq j \leq k-1$,

$$A_{s,kn+j}^{(k)} = (kn+j-s)A_{s,kn+j-1}^{(k)} + (s+1)A_{s+1,kn+j-1}^{(k)} \tag{3}$$

and

$$A_{s,kn+k}^{(k)} = (1+s)A_{s,kn+k-1}^{(k)} + (kn+k-s)A_{s-1,kn+k-1}^{(k)}. \tag{4}$$

There are simple formula for lowest and the highest coefficients in the polynomials $A_{kn+j}^{(k)}$ and we can give direct combinatorial proofs of such formulas. That is, we can prove the following.

Theorem 4. *For* $0 \leq j \leq k-1$, *we have*
$A_{0,kn+j}^{(k)} = ((k-1)n+j)! \prod_{i=0}^{n-1}(j+1+i(k-1))$,
$A_{n,kn+j}^{(k)} = (n(k-1)+j)!(k-1)^n n!$.

By starting with our formulas for $A_{0,kn+j}^{(k)}$ and iterating the recursions (3) and (4) up to get formulas $A_{s,kn+j}^{(k)}$ for any $0 \leq s \leq n$, we can prove the following.

Theorem 5. *For all* $0 \leq j \leq k-1$ *and all* $n \geq 0$, *we have*

$$A_{s,kn+j}^{(k)} = ((k-1)n+j)! \left[\sum_{r=0}^{s}(-1)^{s-r}\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{s-r}\prod_{i=0}^{n-1}(r+1+j+(k-1)i)\right].$$

As a corollary to Theorem 5 we get combinatorial proofs for two special cases of the Saalschütz's identity, which in terms of generalized hypergeometric functions can be written as

$${}_3F_2\left[\begin{matrix} a & b & c \\ d & e & \end{matrix};1\right] = \frac{(d-a)_{|c|}(d-b)_{|c|}}{d_{|c|}(d-a-b)_{|c|}}$$

where $d+e = a+b+c+1$ and c is a negative integer[1] (see [**9**] pages 43 and 126).

Corollary 1. *The following identities hold:*

$$\binom{n}{s}^2 = \sum_{r=0}^{s}(-1)^{s-r}\binom{n+r}{r}^2\binom{2n+1}{s-r};$$

$$\binom{n}{s}\binom{n+1}{s+1} = \sum_{r=0}^{s}(-1)^{s-r}\binom{n+r+1}{r}\binom{n+r+1}{r+1}\binom{2n+2}{s-r}.$$

We can get a second set of formulas for $A_{s,kn+j}^{(k)}$ by starting with our formulas for $A_{n,kn+j}^{(k)}$ and iterating the recursions (3) and (4) down to get formulas $A_{s,kn+j}^{(k)}$ for any $0 \leq s \leq n$.

Theorem 6. *For all* $0 \leq j \leq k-1$ *and* $0 \leq s \leq n$,

$$A_{n-s,kn+j}^{(k)} = ((k-1)n+j)! \left[\sum_{r=0}^{s}(-1)^{s-r}\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{s-r}\prod_{i=1}^{n}(r+(k-1)i)\right] \tag{5}$$

[1]For the first identity in Corollary 1, $a = n+1$, $b = n+1$, $c = -s$, $d = 1$, and $e = 2n+2-s$; for the second identity there, $a = n+2$, $b = n+2$, $c = -s$, $d = 2$, and $e = 2n+3-s$

Corollary 2. *For all* $0 \leq j \leq k-1$ *and* $0 \leq s \leq n$,

$$\sum_{r=0}^{s}(-1)^{s-r}\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{s-r}\prod_{i=1}^{n}(r+(k-1)i) =$$
$$\sum_{r=0}^{n-s}(-1)^{n-s-r}\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{n-s-r}\prod_{i=0}^{n-1}(r+1+j+(k-1)i).$$

For example, the $s=0$ of Corollary 2, gives the following identities where $0 \leq j \leq k-1$:

$$(k-1)^n(n!) = \sum_{r=0}^{n}(-1)^{n-r}\binom{(k-1)n+j+r}{r}\binom{kn+j+1}{n-r}\prod_{i=0}^{n-1}(r+1+j+(k-1)i).$$

3. Properties of $B_n^{(k)}(x,z)$

In a manner that is similar to the way that we derived formulas for the coefficients of $A_n^{(k)}(x)$, we can derive closed formulas for the coefficients of $B_n^{(k)}(x)$ and $B_n^{(k)}(x,z)$. Due to space limitations, we will simply give a list of the theorems that we proved about the the coefficients of $B_n^{(k)}(x)$ and $B_n^{(k)}(x,z)$.

For $0 \leq j \leq k-2$, let Θ_{kn+j} be the operator that sends z^0x^s to $(1+s+(k-1)n+j)z^0x^s+(n-s)z^0x^{s+1}$ and z^1x^s to $(1+s+(k-1)n+j)z^1x^s+(n-s-1)z^1x^{s+1}+z^0x^{s+1}$. Also let Ψ_{kn+k-1} be the operator that sends z^0x^s to $(s+(k-1)(n+1))z^0x^s+z^1x^s+(n-s)z^0x^{s+1}$ and z^1x^s to $(1+s+(k-1)(n+1))z^1x^s+(n-s)z^1x^{s+1}$. Then we have the following.

Theorem 7. *For any* $k \geq 2$ *and* $n \geq 0$,

(1) $B_1^{(k)}(x,z) = 1$,
(2) $B_{kn+j+1}^{(k)}(x,z) = \Theta_{kn+j}(B_{kn+j}^{(k)}(x,z))$ *for* $0 \leq j \leq k-2$, *and*
(3) $B_{kn+k}^{(k)}(x,z) = \Psi_{kn+k-1}(B_{kn+k-1}^{(k)}(x,z))$.

Theorem 8. *For all* $n \geq 0$, $k \geq 2$, *and* $0 \leq j \leq k-1$,

(1) $B_{0,kn+j}^{(k)} = ((k-1)n+j)!\prod_{i=1}^{n}(1+(k-1)i)$,
(2) $B_{0,0,kn+j}^{(k)} = ((k-1)n+j)!\prod_{i=1}^{n}((k-1)i) = ((k-1)n+j)!(k-1)^n(n!)$, *and*
(3) $B_{1,0,kn+j}^{(k)} = ((k-1)n+j)!\,((\prod_{i=1}^{n}(1+(k-1)i)) - (k-1)^n(n!))$.

Theorem 9. *We have* $B_{n,kn}^{(k)} = B_{0,n,kn}^{(k)} = 0$, *and for all* $n \geq 0$, $k \geq 2$, *and* $1 \leq j \leq k-1$,

(1) $B_{n,kn+j}^{(k)} = ((k-1)n+j)!\prod_{i=0}^{n-1}(j+(k-1)i)$,
(2) $B_{0,n,kn+j}^{(k)} = ((k-1)n+j)!\prod_{i=0}^{n-1}(j+(k-1)i)$, *and*
(3) $B_{1,n,kn+j}^{(k)} = 0$.

Theorem 10. *For all* $n \geq 0$, $k \geq 2$, *and* $0 \leq j \leq k-1$,

$$B_{1,n-1,kn+j}^{(k)} = ((k-1)n+j)!\sum_{p=0}^{n-1}\left(\prod_{i=0}^{p-1}(j+(k-1)i)\right)\left(\prod_{i=p+1}^{n-1}(1+j+(k-1)i)\right). \tag{6}$$

We let $\Omega(k,n,j) = \sum_{p=0}^{n-1}\left(\prod_{i=0}^{p-1}(j+(k-1)i)\right)\left(\prod_{i=p+1}^{n-1}(1+j+(k-1)i)\right)$.

Lemma 11. *For all* $n \geq 1$, $k \geq 2$, *and* $r \geq 0$,

$$\begin{aligned}\Omega(k,n,r) &= \sum_{p=0}^{n-1}\left(\prod_{i=0}^{p-1}(r+(k-1)i)\right)\left(\prod_{i=p+1}^{n-1}(1+r+(k-1)i)\right)\\ &= \left(\prod_{i=0}^{n-1}(1+r+(k-1)i)\right)-\left(\prod_{i=0}^{n-1}(r+(k-1)i)\right).\end{aligned} \tag{7}$$

Next we turn to general formulas for $B_{s,kn+j}^{(k)}$, $B_{0,s,kn+j}^{(k)}$ and $B_{1,s,kn+j}^{(k)}$.

THEOREM 12. *For all $n \geq 0$, $k \geq 2$, and $0 \leq j \leq k-1$,*

$$B^{(k)}_{n-s,kn+j} = ((k-1)n+j)! \sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{s-r} \prod_{i=0}^{n-1} (r+j+(k-1)i). \tag{8}$$

As a corollary to Theorem 12 we get combinatorial proofs for two special cases of the Saalschütz's identity [2].

COROLLARY 3. *The following identities hold:*

$$\frac{n+1}{s+1}\binom{n}{s}^2 = \sum_{r=0}^{n-s} (-1)^{n-s-r} \binom{n+r}{r}\binom{n+r+1}{r}\binom{2n+2}{n-s-r};$$

$$\binom{n-1}{s}\binom{n+1}{s+1} = \sum_{r=0}^{n-s} (-1)^{n-s-r} \binom{n+r}{r}\binom{n+r-1}{r-1}\binom{2n+1}{n-s-r}.$$

THEOREM 13. *For all $n \geq 0$, $k \geq 2$, and $0 \leq j \leq k-1$,*

$$B^{(k)}_{1,n-1-s,kn+j} = ((k-1)n+j)! \sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j}{s-r} \Omega(k,n,r+j). \tag{9}$$

where $\Omega(k,n,r)$ is given by (7).

THEOREM 14. *For all $n \geq 0$, $k \geq 2$, and $0 \leq j \leq k-1$,*

$B^{(k)}_{0,n,kn+j} = ((k-1)n+j)! \prod_{i=0}^{n-1} (j+(k-1)i)$ *and*

$B^{(k)}_{0,n-1-s,kn+j} =$
$\sum_{r=0}^{s+1} (-1)^{s+1-r} \binom{(k-1)n+j+r}{r} \binom{kn+j}{s+1-r} \prod_{i=0}^{n-1} (r+j+(k-1)i)$
$- \sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+k-1+r}{r} \binom{kn+j}{s-r} \prod_{i=0}^{n-1} (1+r+j+(k-1)i)$ *for $0 \leq s \leq n-1$.*

THEOREM 15. *For all $k \geq 2$, $n \geq 0$, $0 \leq j \leq k-1$, and $0 \leq s \leq n$.*

$$B^{(k)}_{s,kn+j} = ((k-1)n+j)! \left[\sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{s-r} \prod_{i=1}^{n} (1+r+(k-1)i) \right] \tag{10}$$

THEOREM 16. *For all $n \geq 0$, $k \geq 2$, and $0 \leq j \leq k-1$,*

$$\sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{s-r} \prod_{i=1}^{n} (1+r+(k-1)i) = \sum_{r=0}^{n-s} (-1)^{n-s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{n-s-r} \prod_{i=0}^{n-1} (r+j+(k-1)i). \tag{11}$$

4. Bijective proofs related to the context

The results of the previous section led to a number identities among the coefficients $A^{(k)}_{s,n}$, $B^{(k)}_{s,n}$, and $B^{(k)}_{i,s,n}$ for various special values of s and n. This naturally leads one to ask whether there are simple bijective proofs of these identities. In this section, we shall list a number of these identities for which we can give bijective proofs. Our bijections generalize some of the bijective proofs from [**5**].

THEOREM 17. *For all $k \geq 2$ and $n \geq 0$*

$$\begin{aligned} A^{(k)}_{s,kn+k-1} &= A^{(k)}_{n-s,kn+k-1} \text{ for } 0 \leq s \leq n, \\ B^{(k)}_{0,s,kn+k-1} &= B^{(k)}_{0,n-s,kn+k-1} \text{ for } 0 \leq s \leq n, \text{ and} \\ B^{(k)}_{1,s,kn+k-1} &= B^{(k)}_{1,n-1-s,kn+k-1} \text{ for } 0 \leq s \leq n-1. \\ A^{(k)}_{s,kn+k-1} &= B^{(k)}_{s,kn+k-1} \text{ for } 0 \leq s \leq n. \end{aligned}$$

[2]For the first identity in Corollary 3, in the Saalschütz's identity, $a = n+2$, $b = n+1$, $c = s-n$, $d = 1$, and $e = n+s+3$; for the second identity there, $a = n+1$, $b = n$, $c = s-n$, $d = 0$, and $e = n+s+2$

Theorem 18. *For all $k \geq 3$, $n \geq 0$, and $1 \leq j \leq \lfloor n/k \rfloor$,*

$$\begin{aligned} A^{(k)}_{j,kn+k-2} &= B^{(k)}_{0,j,kn+k-2} + B^{(k)}_{1,j-1,kn+k-2}. \\ A^{(k)}_{j,kn+k-2}(x) &= A^{(k)}_{n-j,kn+k-2}(x). \end{aligned}$$

Thus $A^{(k)}_{kn+k-2}(x)$ is symmetric for $n \geq 0$ and $k \geq 3$.

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