INTERVAL AVOIDANCE FOR LENGTH THREE PATTERNS

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ABSTRACT. Recently, A. Yong and the second author introduced the notion of interval pattern avoidance as a language for describing certain local properties of Schubert varieties. In this paper we initiate the systematic study of combinatorial properties of interval pattern avoidance and, in particular, enumerative properties by looking at intervals formed using permutation patterns of length three.

We show that this new notion of pattern avoidance reduces to classical pattern avoidance for every interval (taken with respect to Bruhat order in the symmetric group $\mathfrak{S}_3$) except for the interval $[123, 321]$. We also present some data and ideas for enumerating permutations avoiding $[123, 321]$.

1. INTRODUCTION

Recently, A. Yong and the second author introduced the notion of *interval pattern avoidance* in connection with the study of Schubert varieties. Focusing upon the algebro-geometric ideas behind interval pattern avoidance, they showed that it can be used to characterize Gorensteinness [9] in addition to several other common invariants [10]. In this paper we initiate the systematic combinatorial study of interval pattern avoidance with an emphasis on enumerative properties by looking at intervals formed using patterns of length three.

Lakshmibai and Sandhya [7] characterized smooth Schubert varieties as those associated to permutations avoiding the (classical) patterns 4231 and 3412. The appearance of pattern avoidance in this geometric setting remained a mystery until explained by Billey and Braden [1]. However, when characterizing Schubert varieties which are Gorenstein, A. Yong and the second author found that classical pattern avoidance was insufficient, and so a generalization called *Bruhat-restricted pattern avoidance* was defined for this purpose. Interval pattern avoidance (defined in Section 2 below) is in essence a further generalization of Bruhat-restricted pattern avoidance that was discovered in an attempt to give a geometric explanation of its appearance. As explained in [10], interval pattern avoidance is in some sense universal in characterizing singularities of Schubert varieties.

In a separate work, S. Butler and M. Bousquet-Mélou [4] used Bruhat-restricted pattern avoidance to characterize which Schubert varieties are factorial. In the notation of this paper (see Section 2), these are the Schubert varieties indexed by permutations that avoid both the (classical) pattern 4231 and the interval $[3142, 3412]$ (or equivalently 4231 and $[2413, 3412]$). They were also able to give a generating

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function for enumerating the permutations in the symmetric group $\mathfrak{S}_n$ satisfying this avoidance condition.

In Section 2, we define interval pattern avoidance and give some of its basic properties. This includes an interpretation in terms of the graph of a permutation that was implicit in previous work [2, 5, 6, 8] describing the singular locus of Schubert varieties. In Section 3, we treat all intervals in S_3 except for $[123, 321]$ by showing that they reduce to classical pattern avoidance and hence have avoidance sets enumerated by the Catalan numbers. Finally in Section 4, we describe ongoing work to enumerate permutations avoiding the interval $[123, 321]$.

2. Interval Pattern Avoidance

In order to fix the terminology used for pattern avoidance in this extended abstract, we begin with the following definition of permutation pattern containment:

Definition 2.1. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ and $\pi = \pi_1\pi_2\cdots\pi_m \in \mathfrak{S}_m$ with $m \leq n$. Then an *embedding* of π into σ is a choice of indices $i_1 < i_2 < \cdots < i_m$ such that $\sigma_{i_j} < \sigma_{i_k}$ if and only if $\pi_j < \pi_k$ for each $j, k = 1, 2, \ldots, m$.

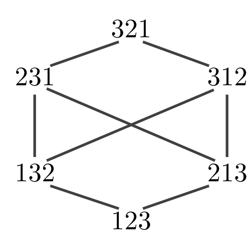
Given an embedding of π into σ, we say that the permutation pattern π *embeds into* (or, equivalently, *is contained in*) the permutation σ. Moreover, if no embeddings of π into σ are possible, then we say that σ avoids π and denote the *avoidance set* of all permutations $\sigma \in \mathfrak{S}_n$ avoiding a pattern π by $S_n(\pi)$. (We refer the reader to [3] for more information about permutation patterns in general.)

Before extending this notion of pattern avoidance, we first recall the definition of the (strong) Bruhat order:

Definition 2.2. Given two permutations $\sigma, \tau \in \mathfrak{S}_n$, we say that $\sigma < \tau$ under the *Bruhat order* if τ can be transformed into σ by successively interchanging any two elements in τ that realize an embedding of the pattern 21.

In other words, $\sigma < \tau$ is a generating relation if σ can be obtained by "undoing" a 21 pattern (a.k.a. an *inversion*) in τ. The Bruhat order $\leq$ on the symmetric group $\mathfrak{S}_n$ is then defined to be the reflexive transitive closure of these generating relations. Equivalently, if we define the *length* $\ell(\sigma)$ of each permutation $\sigma \in \mathfrak{S}_n$ to be the number of inversions in σ, then we can also realize the partial ordering $\leq$ as the transitive closure of the covering relation $\sigma \prec \tau$ defined by the conditions $\sigma = \tau t$ for some transposition t and $\ell(\sigma) = \ell(\tau) + 1$. Denoting $\tau = \tau_1\tau_2\cdots\tau_n$, this corresponds to "undoing" an embedding of 21 at positions $i_1 < i_3$ in τ when there is no index i_2 for which $i_1 < i_2 < i_3$ and $\tau_{i_1} > \tau_{i_2} > \tau_{i_3}$.

As usual, one can visualize the resulting partially ordered set as a Hasse diagram:



Example 2.3. The permutation $1324 < 3412$ under the Bruhat order using the covering relations $1324 \prec 3124 \prec 3142 \prec 3412$.

Given two permutations $\sigma, \tau \in \mathfrak{S}_n$ with $\sigma \leq \tau$, we denote their interval in the Bruhat order by $[\sigma, \tau]$. We can then extend the above notion of embedding to such intervals as follows:

Definition 2.4. Suppose that $\pi \leq \rho \in \mathfrak{S}_m$ and $\sigma \leq \tau \in \mathfrak{S}_n$ with $m \leq n$. Then we say that the interval $[\pi, \rho]$ *embeds* into the interval $[\sigma, \tau]$ if the following conditions are satisfied:

(1) there is a common embedding $i_1 < i_2 < \cdots < i_m$ of π into σ and of ρ into τ
(2) the intervals $[\pi, \rho]$ and $[\sigma, \tau]$ are order-isomorphic.

As before, we say that the interval $[\sigma, \tau]$ *avoids* the interval $[\pi, \rho]$ if there is no embedding of the latter into the former. Note that this reduces to the usual notion of pattern avoidance if we set $\pi = \rho$ and $\sigma = \tau$ since the intervals $[\pi, \rho] = \{\rho\}$ and $[\sigma, \tau] = \{\tau\}$ are trivially order-isomorphic.

Example 2.5.

(1) The interval $[1324, 3412]$ embeds into the interval [4**1325**, 4**3512**] (as the bold, underlined elements) since 41325 contains the pattern 1324, 43512 contains the pattern 3412, and the intervals are order-isomorphic (as one can easily check).
(2) The interval [**1**2**43**5**6**, **4**2**61**5**3**] avoids the interval $[1324, 3412]$ even though 124356 contains the pattern 1324 and 426153 contains the pattern 3412. By computing the lengths of the permutations,

$$\ell(426153) - \ell(124356) = 7 > 3 = \ell(3412) - \ell(1324)$$

so that the intervals cannot be order-isomorphic.

The computation of lengths in Example 2.5(2) suggests the following equivalent formulation of interval pattern embeddings.

Lemma 2.6. *Let $\pi \leq \rho \in \mathfrak{S}_m$ and $\sigma \leq \tau \in \mathfrak{S}_n$ with $m \leq n$. Then, given a common embedding $i_1 < i_2 < \cdots < i_m$ of π into $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ and of ρ into $\tau = \tau_1\tau_2\cdots\tau_n$, the interval $[\pi, \rho]$ embeds into the interval $[\sigma, \tau]$ if and only if the following two conditions are satisfied:*

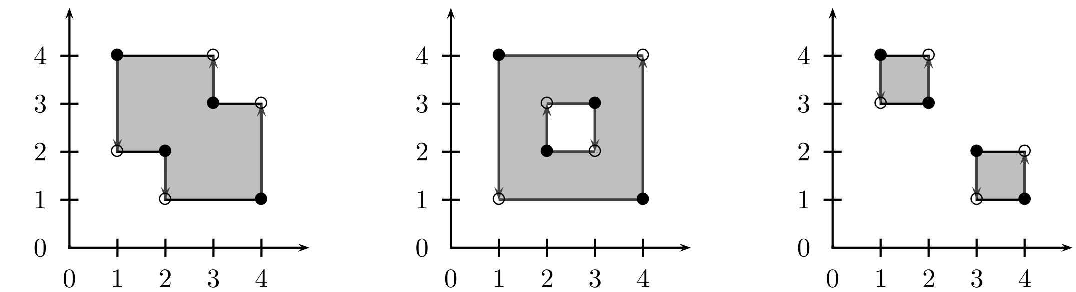
(1) *$\sigma_i = \tau_i$ for each index $i \notin \{i_1, i_2, \ldots, i_m\}$*
(2) *$\ell(\tau) - \ell(\sigma) = \ell(\rho) - \ell(\pi)$.*

In particular, this implies that it suffices to consider only the three permutations π, ρ, and τ in an interval embedding since σ can then be uniquely reconstructed. Moreover, we can then extend the definition of avoidance set from a single pattern to any interval $[\pi, \rho]$ in Bruhat order as follows:

$$S_n([\pi, \rho]) = \{\tau \in \mathfrak{S}_n \mid [\pi, \rho] \text{ does not embed into } [\sigma, \tau] \text{ for any } \sigma \leq \tau\}$$

Example 2.7.

(1) From Example 2.5(1), 43512 contains the interval $[1324, 3412]$ since it embeds into the interval $[41325, 43512]$.
(2) Based upon Example 2.5(2), one can see that $426153 \in S_n([1324, 3412])$.



(a) The forbidden region for the interval $[2143, 4231]$.

(b) The forbidden region for the interval $[1324, 4231]$.

(c) The forbidden region for the interval $[3412, 4321]$.

FIGURE 2.1. Examples of forbidden regions formed by intervals taken under Bruhat order.

We close this section by giving an interpretation of interval pattern avoidance in terms of the graphs of the two permutation defining a interval, which imply a certain "forbidden" region. Specifically, given $\pi = \pi_1\pi_2\cdots\pi_m$ and $\rho = \rho_1\rho_2\cdots\rho_m$, graph the points (i, π_i) as circles and (i, ρ_i) as solid dots in the Euclidean plane $\mathbb{R}^2$. Now connect points having the same ordinate by plain lines, and connect points having the same abscissa by lines with arrows pointing toward the circles. Then form a (possibly disconnected) *forbidden region* by shading each smallest bounded region enclosed by plain lines and adjacent directed lines with downward-directed lines along the left-most boundaries and upward-directed lines along the right-most boundaries. Several examples are given in Figure 2.1.

It should be clear that a permutation τ contains the interval $[\pi, \rho]$ if the forbidden region induced by embedding ρ into τ contains no points. This is illustrated in the analysis given in Section 3 below.

3. Short Intervals and Catalan Numbers

Now we study the interval pattern avoidance for intervals in $\mathfrak{S}_3$. Since inverses and reverse complements give us equivalences, we see that

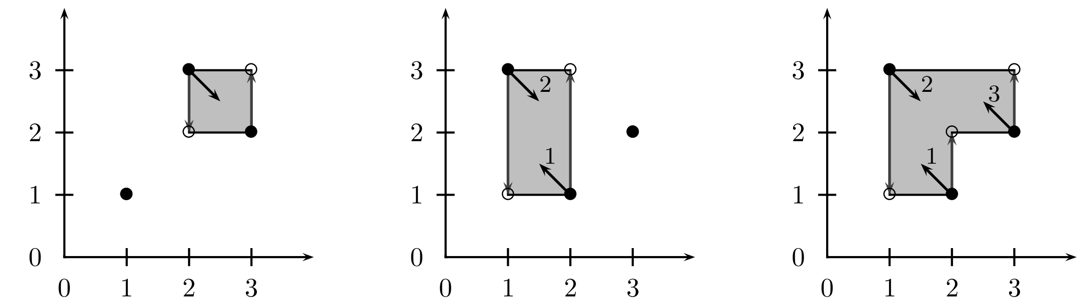
$$\begin{aligned}
\#S_n([123, 132]) &= \#S_n([123, 213]),\\
\#S_n([123, 231]) &= \#S_n([123, 312]),\\
\#S_n([132, 321]) &= \#S_n([213, 321])\\
\#S_n([312, 321]) &= \#S_n([231, 321]), \text{ and}
\end{aligned}$$

$$\#S_n([132, 312]) = \#S_n([213, 231]) = \#S_n([213, 312]) = \#S_n([132, 231]).$$

From the graphical description, we see that taking reverses and complements also preserves the sizes of interval avoidance sets, remembering that the lower end of the interval has become the upper and vice versa. Therefore,

$$\#S_n([123, 132]) = \#S_n([231, 321]), \text{ and } \#S_n([123, 132]) = \#S_n([231, 321]).$$

This analysis shows that, other than the classical avoidance classes, we have four potentially new classes. In the remainder of this section, we reduce three of these avoidance set, namely $S_n([123, 132])$, $S_n([123, 312])$, and $S_n([132, 312])$, to classical pattern avoidance. Hence these classes are counted by Catalan numbers.



(a) Forbidden region reduction for $[123, 132]$.

(b) Forbidden region reduction for $[132, 312]$.

(c) Forbidden region reduction for $[123, 312]$.

FIGURE 3.1. Forbidden region reductions for the three classes of intervals in $\mathfrak{S}_n$ defined by covering relations.

The analysis of the class $S_n([123, 321])$ is ongoing work and will be described in the next section.

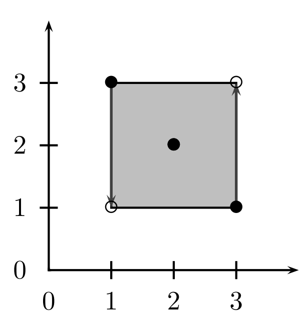
We now show that $S_n([123, 132]) = S_n(132)$. Clearly, every permutation avoiding 132 avoids $[123, 132]$. It therefore remains to be shown that every permutation w with an embedding of 132 contains an embedding of $[123, 132]$. Let i_1, i_2, i_3 be an embedding of 132. Now let i'_2 be the largest index with $i_2 \le i'_2 < i_3$ such that $w_{i'_2} > w_{i_3}$. Then i_1, i'_2, i_3 will be an embedding of 132, and $\ell(w \cdot (i'_2 \leftrightarrow i_3)) = \ell(w) - 1$, since, by construction, there does not exist k such that $i'_2 < k < i_3$ and $w_{i'_2} > w_k > w_{i_3}$. (Here $(i'_2 \leftrightarrow i_3)$ denotes the transposition interchanging i'_2 and i_3.) This argument is summarized by Figure 3.1(a).

The other two cases are similar as summarized in Figure 3.1(b,c). For a permutation with an embedding i_1, i_2, i_3 of 312, we can choose i'_2 minimal so that $i_1 < i'_2$ and $w_{i'_2} < w_{i_3}$, then choose i'_1 maximal so that $i'_1 < i'_2$ and $w_{i'_1} > w_{i_3}$. Then i'_1, i'_2, i_3 is an embedding of $[132, 312]$. To get an embedding of $[123, 312]$, we can now choose i'_3 minimal such that $i'_3 > i'_2$ and $w_{i'_3} < w_{i'_1}$.

One might naively guess that similar reductions apply to all intervals of length one, but this is not the case. For example, 53124 contains an embedding of 4123 but is nonetheless in $S_n([1423, 4123])$.

4. The Interval $[123, 321]$

Given the analysis in Section 3 above, the last remaining case in analyzing interval avoidance for patterns in $\mathfrak{S}_3$ is the single interval $[123, 321]$ of length three. This interval generates the following forbidden region:



We have been able to compute the value of $\#S_n([123, 321])$ for $n = 1, 2, \ldots, 12$ as

$$1, 2, 5, 15, 51, 194, 810, 3675, 17935, 93481, 517129, 3021133.$$

However, our attempts at providing a combinatorial interpretation for this sequence have so far been unsuccessful. Since it clear that $S_n(321) \subset S_n([123, 321])$, our current efforts involve trying to characterize permutations containing the pattern 321 that are also in $S_n([123, 321])$. The first few values of $\#(S_n([123, 321]) \backslash S_n(321))$ for $n = 4, \ldots, 12$ are

$$1, 9, 62, 381, 2245, 13073, 76685, 458343, 2813121.$$

One possibility for trying to directly count these permutations is to filter the avoidance set $S_n([123, 321])$ according to the largest m for which the permutation has an embedding of $m23 \cdots (m-1)1$, or perhaps using a finer filtration using more patterns. It is not difficult to see that there are $4m - 8$ distinct patterns in $S_{m+1}([123, 321])$ containing $m23 \cdots (m-1)1$, but we have not yet been able to push this analysis through. We have also found necessary and sufficient conditions describing when no left-to-right minimum is involved in an embedding of $[123, 321]$, but this places significant restrictions on the further structure of the permutation, making a recursive analysis appear quite difficult.

References

[1] S. Billey and T. Braden. "Lower bounds for Kazhdan-Lusztig polynomials from patterns." *Transform. Groups* **8** (2003), 321–332.

[2] S. Billey and G. Warrington. "Maximal singular loci of Schubert varieties in $SL(n)/B$." *Trans. Amer. Math. Soc.* **335** (2003), 3915–3945.

[3] M. Bóna. *Combinatorics of Permutations*. Chapman & Hall/CRC Press, 2004.

[4] M. Bousquet-Mélou and S. Butler. "Forest-like permutations." Preprint, 2006. arXiv:math.CO/0603617.

[5] A. Cortez. "Singularités génériques et quasi-résolutions des variétés de Schubert pour le groupe linéaire." *Adv. Math.* **178** (2003), 396–445.

[6] C. Kassel, A. Lascoux and C. Reutenauer. "The singular locus of a Schubert variety." *J. Algebra* **269** (2003), 74–108.

[7] V. Lakshmibai and B. Sandhya. "Criterion for smoothness of Schubert varieties in $SL(n)/B$." *Proc. Indian Acad. Sci. Math. Sci.* **100** (1990), 45–52.

[8] L. Manivel. "Le lieu singulier des variétés de Schubert." *Internat. Math. Res. Notices* **16** (2001), 849–871.

[9] A. Woo and A. Yong. "When is a Schubert variety Gorenstein?" Accepted by *Adv. in Math.*, 2005. arXiv:math.CO/0409490.

[10] A. Woo and A. Yong. "Governing Singularities of Schubert Varieties." Preprint, 2006. arXiv:math.AG/0603273.

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