Permutation Patterns 2006
Reykjavik University, Reykjavik, Iceland

q-Analogues of formulas for the number of ascents and descents with specified equivalences mod k

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ABSTRACT. Given a sequence of integers of length j, τ, we say that a permutation σ has a τ-k-match starting at position i, if the elements in position $i, i+1, \ldots, i+j-1$ in σ have the same relative order as the elements of τ and belong to the same equivalence class mod k, element by element, for some $k \geq 2$. If Υ is set of sequences of length j, then we say that a permutation σ has an Υ-k-match starting at position j if it has a τ-k-match at position j for some $\tau \in \Upsilon$. Some recent papers have studied the distribution of τ-k-matches and Υ-k-matches in permutations. In this paper, we provide q-analogues to many previous formulas proved in this area.

1. Introduction

Given any sequence $\sigma = \sigma_1 \cdots \sigma_n$ of distinct integers, we let $red(\sigma)$ be the permutation that results by replacing the i-th largest integer that appears in the sequence σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $red(\sigma) = 1\ 4\ 3\ 2$. Given a permutation τ in the symmetric group S_j, we define a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ to have a τ-match at place i provided $red(\sigma_i \cdots \sigma_{i+j-1}) = \tau$. Let τ-$mch(\sigma)$ be the number of τ-matches in the permutation σ. To prevent confusion, we note that a permutation not having a τ-match is different than a permutation being τ-avoiding. A permutation is called τ-avoiding if there are no indices $i_1 < \cdots < i_j$ such that $red[\sigma_{i_1} \cdots \sigma_{i_j}] = \tau$. For example, if $\tau = 2\ 1\ 4\ 3$, then the permutation $3\ 2\ 1\ 4\ 6\ 5$ does not have a τ-match but it does not avoid τ since $red[2\ 1\ 6\ 5] = \tau$.

In the case where $|\tau| = 2$, then τ-$mch(\sigma)$ reduces to familiar permutation statistics. That is, if $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let $Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$ and $Rise(\sigma) = \{i : \sigma_i < \sigma_{i+1}\}$. Then it is easy to see that $(2\ 1)$-$mch(\sigma) = des(\sigma) = |Des(\sigma)|$ and $(1\ 2)$-$mch(\sigma) = rise(\sigma) = |Rise(\sigma)|$.

A number of recent publications have analyzed the distribution of τ-matches in permutations. See, for example, [**EN03, Kit03, Kit**]. A number of interesting results have been proved. For example, let τ-nlap(σ) be the maximum number of non-overlapping τ-matches in σ where two τ-matches are said to overlap if they contain any of the same integers. Then Kitaev [**Kit03, Kit**] proved the following.

THEOREM 1.1.

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\sigma)} = \frac{A(t)}{(1-x)+x(1-t)A(t)} \tag{1.1}$$

where $A(t) = \sum_{n\geq 0} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}|$.

In other words, if the exponential generating function for the number of permutations in S_n without any τ-matches is known, then so is the exponential generating function for the entire distribution of the statistic τ-nlap.

Mendes and Remmel [**MR05**] proved a number of extensions of Kitaev's result. For example, suppose $\Upsilon \subseteq S_j$. We say that a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has an Υ-match at place i provided

2000 *Mathematics Subject Classification.* Primary 17B10; Secondary 05E10.
Key words and phrases. ascents, descents, parity, pattern matching, permutations, q-analogues.
The second author is supported in part by NSF grant DMS 0400507.

$red[\sigma_i \cdots \sigma_{i+j-1}] \in \Upsilon$. Let Υ-mch(σ) and Υ-nlap(σ) be the number of Υ-matches and non-overlapping Υ matches in σ, respectively.

THEOREM 1.2. [**MR05**]

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]!} \sum_{\sigma \in S_n} x^{\Upsilon-nlap(\sigma)} q^{inv(\sigma)} = \frac{A_q^{\Upsilon}(t)}{(1-x)+x(1-t)A_q^{\Upsilon}(t)}. \tag{1.2}$$

where $A_q^{\Upsilon}(t) = \sum_{n \geq 0} \frac{t^n}{[n]!} \sum_{\sigma \in S_n : \Upsilon\text{-mch}(\sigma)=0} q^{inv(\sigma)}$.

More recently, a number of papers [**KR05, KR06, L06**] have considered a more refined pattern matching condition where we take into account conditions involving equivalence mod k for some integer $k \geq 2$. That is, suppose we fix $k \geq 2$ and we are given some sequence of distinct integers $\tau = \tau_1 \cdots \tau_j$. Then we say that a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has a τ-k-equivalence match at place i provided $red(\sigma_i \cdots \sigma_{i+j-1}) = red(\tau)$ and for all $s \in \{0, \ldots, j-1\}$, $\sigma_{i+s} = \tau_{1+s} \mod k$. For example, if $\tau = 1\ 2$ and $\sigma = 5\ 1\ 7\ 4\ 3\ 6\ 8\ 2$, then σ has τ-matches starting at positions 2, 5, and 6. However, if $k = 2$, then only the τ-match starting at position 5 is a τ-2-equivalence match. Later, it will be explained that the τ-match starting a position 2 is a (1 3)-2-equivalence match and the τ-match starting a position 6 is a (2 4)-2-equivalence match. Let τ-k-$emch(\sigma)$ be the number of τ-k-equivalence matches in the permutation σ. Let τ-k-$enlap(\sigma)$ be the maximum number of non-overlapping τ-k-equivalence matches in σ where two τ-matches are said to overlap if they contain any of the same integers.

More generally, if Υ is a set of sequences of distinct integers of length j, then we say that a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has a Υ-k-equivalence match at place i provided there is a $\tau \in \Upsilon$ such that $red(\sigma_i \cdots \sigma_{i+j-1}) = red(\tau)$ and for all $s \in \{0, \ldots, j-1\}$, $\sigma_{i+s} = \tau_{1+s} \mod k$. Let Υ-k-$emch(\sigma)$ be the number of Υ-k-equivalence matches in the permutation σ and Υ-k-$enlap(\sigma)$ be the maximum number of non-overlapping Υ-k-equivalence matches in σ.

Then one can study the polynomials

$$T_{\tau,k,n}(x) = \sum_{\sigma \in S_n} x^{\tau\text{-}k\text{-}emch(\sigma)} = \sum_{s=0}^{n} T_{\tau,k,n}^{s} x^s \text{ and} \tag{1.3}$$

$$U_{\Upsilon,k,n}(x) = \sum_{\sigma \in S_n} x^{\Upsilon\text{-}k\text{-}emch(\sigma)} = \sum_{s=0}^{n} U_{\Upsilon,k,n}^{s} x^s. \tag{1.4}$$

In particular, [**KR05, KR06, L06**] focused on certain special cases of these polynomials where we consider only patterns of length 2. That is, fix $k \geq 2$ and let A_k equal the set of all sequences $(a\ b)$ such that $1 \leq a < b \leq 2k$ where there is no lexicographically smaller sequence $x\ y$ having the property that $x \equiv a \mod k$ and $y \equiv b \mod k$. For example,

$$A_4 = \{1\ 2, 1\ 3, 1\ 4, 1\ 5, 2\ 3, 2\ 4, 2\ 5, 2\ 6, 3\ 4, 3\ 5, 3\ 6, 3\ 7, 4\ 5, 4\ 6, 4\ 7, 4\ 8\}.$$

Let $D_k = \{b\ a : a\ b \in A_k\}$ and $E_k = A_k \cup D_k$. Thus E_k consists of all k-equivalence patterns of length 2 that we could possibly consider. Note that if $\Upsilon = A_k$, then Υ-k-$emch(\sigma) = rise(\sigma)$ and if $\Upsilon = D_k$, then Υ-k-$emch(\sigma) = des(\sigma)$.

Kitaev and Remmel [**KR05, KR06**] found explicit formulas for the coefficients $U_{\Upsilon,k,n}^{s}$ in certain special cases. In particular, they studied descents according to the equivalence class mod k of either the first or second element in a descent pair. That is, for any set $X \subseteq \{0, 1, 2, \ldots\}$, define

- $\overleftarrow{Des}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1}\ \&\ \sigma_i \in X\}$ and $\overleftarrow{des}_X(\sigma) = |\overleftarrow{Des}_X(\sigma)|$
- $\overrightarrow{Des}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1}\ \&\ \sigma_{i+1} \in X\}$ and $\overrightarrow{des}_X(\sigma) = |\overrightarrow{Des}_X(\sigma)|$

In [**KR06**], Kitaev and Remmel studied the polynomials

(1) $A_n^{(k)}(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{kN}(\sigma)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} A_{j,n}^{(k)} x^j$ and
(2) $B_n^{(k)}(x, z) = \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{kN}(\sigma)} z^{\chi(\sigma_1 \in kN)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \sum_{i=0}^{1} B_{i,j,n}^{(k)} z^i x^j$.

where $kN = \{0, k, 2k, \ldots\}$. Again both $A_n^{(k)}(x)$ and $B_n^{(k)}(x, z)$ are special cases of $U_{\Upsilon,k,n}(x)$. Then, for example, Kitaev and Remmel [**KR06**] proved there are two different closed formulas for the coefficients $A_{s,kn+j}^{(k)}$.

THEOREM 1.3. *For all $0 \leq j \leq k-1$ and all $n \geq 0$, we have*

$$\begin{aligned} A^{(k)}_{s,kn+j} &= ((k-1)n+j)! \sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{s-r} \prod_{i=0}^{n-1} (r+1+j+(k-1)i) \\ &= ((k-1)n+j)! \sum_{r=0}^{n-s} (-1)^{n-s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{n-s-r} \prod_{i=1}^{n} (r+(k-1)i) \end{aligned}$$

Liese [**L06**] extended these results as follows. Let $k \geq 2$ and $\Upsilon = \{(x_1, y_1), (x_2, y_2), \ldots, (x_t, y_t)\}$ be a subset of A_k such that for all i, j $y_i \equiv y_j \mod k$. Then we define $y = min(\{y_1, \ldots, y_t\})$ and $\alpha = |\{x_i : x_i < y\}|$. We then let $Asc_{\Upsilon,k}(\sigma) = \{i : \sigma_i < \sigma_{i+1} \ \& \ \sigma_i \equiv x_j \mod k \ \& \ \sigma_{i+1} \equiv y_j \mod k \text{ for some } (x_j, y_j) \in \Upsilon\}$. We shall call the elements of $Asc_{\Upsilon,k}(\sigma)$ the Υ-ascents of σ and we let $asc_{\Upsilon,k}(\sigma) = |Asc_{\Upsilon,k}(\sigma)|$. Then for all $n \geq 0$ and $j \in \{0, \ldots, k-1\}$, we define

$$U_{\Upsilon,k,kn+j}(x) = \sum_{\sigma \in S_{kn+j}} x^{asc_{\Upsilon,k}(\sigma)} = \sum_{s=0}^{n} U^s_{\Upsilon,k,kn+j} x^s. \tag{1.5}$$

Then Liese [**L06**], proved the following.

THEOREM 1.4. *For all $y-k \leq j \leq y-1$ and all $n \geq s$ such that $kn+j > 0$, we have*

$$U^s_{\Upsilon,k,kn+j} = ((k-1)n+j)! \left[\sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+r+j}{r} \binom{kn+j+1}{s-r} \Gamma(r,j,n) \right]$$

$$U^s_{\Upsilon,k,kn+j} = ((k-1)n+j)! \left[\sum_{r=0}^{n-s} (-1)^{n-s-r} \binom{(k-1)n+r+j}{r} \binom{kn+j+1}{n-s-r} \Omega(r,n) \right]$$

where $\Gamma(r,j,n) = \prod_{i=0}^{n-1} ((k-1)n+r+j+1-\alpha-i(|\Upsilon|-1))$ *and* $\Omega(r,n) = \prod_{i=0}^{n-1} (r+\alpha+i(|\Upsilon|-1))$.

2. q-Analogues

While we have been able to provide q-analogues for all of the above mentioned formulas, we will first take a look at one of the simplest formulas and then one of the more general formulas. In the case where $k = 2$ and $X = E = \{0, 2, 4, \ldots\}$ is the set of even numbers, Kitaev and Remmel [**KR05**] proved that the formulas for $A^{(2)}_n(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_E}$ simplify considerably. That is, if we let $R_n(x) = A^{(2)}_n(x) = \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} R_{s,n} x^s$. Let Δ_{2n+1} be the operator which sends x^s to $sx^{s-1} + (2n-s+1)x^s$ and Γ_{2n+2} be the operator that sends x^s to $(s+1)x^s + (2n-s+1)x^{s+1}$. Then Kitaev and Remmel [**KR05**] proved the following.

THEOREM 2.1. *The polynomials $R_n(x)_{n \geq 1}$ satisfy the following recursions.*

(1) $R_1(x) = 1$,
(2) $R_{2n+1}(x) = \Delta_{2n+1}(R_{2n}(x))$, *and*
(3) $R_{2n+2}(x) = \Gamma_{2n+2}(R_{2n+1}(x))$.

This fact gave rise to the following recursions for $R_{s,n}(x)$.

$$R_{s,2n+1} = (s+1)R_{s+1,2n} + (2n-s+1)R_{s,2n} \tag{2.1}$$

$$R_{s,2n+2} = (s+1)R_{s,2n+1} + (2n-s+2)R_{s-1,2n+1} \tag{2.2}$$

It was through these recursions that Kitaev and Remmel were able to show that

$$R_{k,2n} = \binom{n}{k}^2 (n!)^2, \text{ and} \tag{2.3}$$

$$R_{k,2n+1} = \frac{1}{k+1} \binom{n}{k}^2 ((n+1)!)^2. \tag{2.4}$$

To prove q-analogues of the results, we introduce q-analogues of the operators Δ and Γ. Let $[n]_q = 1+q+\cdot+q^{n-1}$, $[n]_q! = [1]_q \cdots [n]_q$, and $\left[{n \atop k}\right]_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$. Then we let Δ^q_{2n+1} be the operator which sends x^s to

$[s]_q\, x^{s-1} + q^s\, [2n-s+1]_q\, x^s$ and Γ^q_{2n+2} be the operator that sends x^s to $[s+1]_q\, x^s + q^{s+1}\, [2n-s+1]_q\, x^{s+1}$. Next we define q-analogues of the polynomials $R_n(x)$, $R^q_n(x)_{n\geq 1} = \sum_{s=0}^{n} R^q_{s,n} x^s$, via the following recursions.

(1) $R^q_1(x,q) = 1$,
(2) $R^q_{2n+1}(x,q) = \Delta^q_{2n+1}(R^q_{2n}(x))$, and
(3) $R^q_{2n+2}(x,q) = \Gamma^q_{2n+2}(R^q_{2n+1}(x))$.

This fact gives rise to the following recursions for the coefficients $R^q_{s,n}(x)$.

$$R^q_{s,2n+1} = [s+1]_q\, R^q_{s+1,2n} + q^s\, [2n-s+1]_q\, R^q_{s,2n} \tag{2.5}$$

$$R^q_{s,2n+2} = [s+1]_q\, R^q_{s,2n+1} + q^s\, [2n-s+2]_q\, R^q_{s-1,2n+1} \tag{2.6}$$

We can then show that the solution to these recursions are

THEOREM 2.2.

$$\begin{aligned} R^q_{k,2n} &= q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 ([n]_q!)^2, \text{ and} \\ R^q_{k,2n+1} &= \frac{q^{\binom{k+1}{2}}}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 ([n+1]_q!)^2. \end{aligned}$$

The operators Δ^q_{2n+1} and Γ^q_{2n+2} give rise to a natural q-statistic on permutations which is similar to the major index statistic. For any permutation $\sigma \in S_n$, $maj(\sigma) = \sum_{i \in Des(\sigma)} i$. Foata [**F68**] showed that the maj statistic satisfies some simple recursions. That is, for any permutation $\tau = \tau_1 \ldots \tau_{n-1} \in S_{n-1}$, we label the spaces where we can insert n into τ to get a permutation in S_n as follows.

(1) Label the space following τ_{n-1} with 0.
(2) Next label the spaces that lie between descents $\tau_i > \tau_{i+1}$ from right to left with the integers $1, \ldots, des(\tau)$.
(3) Finally label the remaining spaces from left to right with the integers $des(\tau)+1, \ldots, n$.

Thus, for example, if $\tau = 3\ 9\ 2\ 8\ 5\ 4\ 1\ 6\ 7$, then spaces are labeled as follows:

$$\underline{\ }_5 3 \underline{\ }_6 9 \underline{\ }_4 2 \underline{\ }_7 8 \underline{\ }_3 5 \underline{\ }_2 4 \underline{\ }_1 1 \underline{\ }_8 6 \underline{\ }_9 7 \underline{\ }_0.$$

Then Foata proved that if $\tau^{(i)}$ is the result of inserting n into the space labeled i, then for all $i \in \{0, \ldots, n\}$,

$$maj(\tau^{(i)}) = i + maj(\tau).$$

In our case, we can use a similar labeling procedure to define a statistic $Emaj$ such that $R^q_n(x,q) = \sum_{\sigma \in S_n} q^{Emaj(\sigma)} x^{\overleftarrow{des}_E}$. Given any permutation $\sigma = \sigma_1 \cdots \sigma_{2n} \in S_{2n}$ where $\overleftarrow{des}_E(\sigma) = s$, we first label the possible spaces where we can insert $2n+1$ to get a permutation in S_{2n+1} such that $\overleftarrow{des}_E(\sigma) = s-1$ with the integers from 0 to $s-1$ from right to left. These are precisely the spaces that lie between descents $\sigma_i > \sigma_{i+1}$ where σ_i is even. We then label the remaining spaces from left to right with the integers from s to $2n$. If $\sigma = \sigma_1 \cdots \sigma_{2n+1} \in S_{2n+1}$ where $\overleftarrow{des}_E(\sigma) = s$, we first label the spaces such that we can insert $2n+2$ to get a permutation in S_{2n+2} where $\overleftarrow{des}_E(\sigma) = s$ with the integers from 0 to s from right to left. In this case, the first such space is the space following σ_{2n+1} and the remaining spaces are the spaces that lie between descents $\sigma_i > \sigma_{i+1}$ where σ_i is even. We then label the remaining spaces from left to right with the integers from $s+1$ to $2n+1$. We will call this labeling the E-canonical labeling of σ. For example, suppose $\sigma = 3\ 9\ 2\ 8\ 5\ 4\ 1\ 6\ 7$. Then the E-canonical labeling of σ is

$$\underline{\ }_3 3 \underline{\ }_4 9 \underline{\ }_5 2 \underline{\ }_6 8 \underline{\ }_2 5 \underline{\ }_7 4 \underline{\ }_1 1 \underline{\ }_8 6 \underline{\ }_9 7 \underline{\ }_0.$$

Given $\sigma \in S_{n-1}$, we define $(\sigma \downarrow k)$ to be the permutation in S_n that is obtained from σ by placing n in the position that was labeled with k under the E-canonical labeling.

Given $\sigma \in S_n$ where $\sigma = (\tau \downarrow k)$ for some $\tau \in S_{n-1}$, we now recursively define $Emaj(\sigma)$ as follows.

(1) If $n = 1, Emaj(\sigma) = 0$.
(2) If $n > 1, Emaj(\sigma) = k + Emaj(\tau)$.

With this labeling scheme, it is clear that

$$\sum_{j=0}^{n+1} q^{Emaj((\sigma\downarrow j))} = [n+1]\, q^{Emaj(\sigma)}$$

and thus

$$\sum_{\sigma\in S_n} q^{Emaj(\sigma)} = [n]!.$$

In other words, the statistic is Mahonian. Then we can easily prove that

Theorem 2.3.

$$R_n^q(x) = \sum_{\sigma\in S_n} x^{\overleftarrow{des}_E(\sigma)} q^{Emaj(\sigma)} = \sum_{k=0}^{n} R_{k,n}^q x^k.$$

We will now turn to a more general q-analogue. Let us examine the q-analogue of the polynomials $U_{\{(1k)\},k,n}(x)$. For $j=1,\dots,k-1$, let Δ_{kn+j} be the operator which sends x^s to $sx^{s-1}+(kn+j-s)x^s$ and Γ_{kn+k} be the operator that sends x^s to $((k-1)n+k+s-1)x^s+(n-s+1)x^{s+1}$. Then one can easily show that the polynomials $U_{\{(1k)\},k,n}(x)$ are defined by the following recursions.

(1) $U_{\{(1k)\},k,1}(x)=1$,
(2) For $j=1,\dots,k-1$, $U_{\{(1k)\},k,kn+j}(x)=\Delta_{kn+j}(U_{\{(1k)\},k,kn+j-1}(x))$, and
(3) $U_{\{(1k)\},k,kn+k}(x)=\Gamma_{kn+k}(U_{\{(1k)\},k,kn+k-1}(x))$.

We then define q-analogue these recursions as follows. For $j=0,\dots,k-2$ let Δ_{kn+j}^q be the operator which sends x^s to $[s]_q x^{s-1}+q^s[kn+j-s]_q x^s$ and Γ_{kn+k} be the operator that sends x^s to $[(k-1)n+k+s-1]_q x^s + q^{(k-1)n+k+s-1}[n-s+1]x^{s+1}$. Then we define q-analogues of the polynomials $U_{\{(1k)\},k,n}(x)$ by the following recursions.

(1) $U^q_{\{(1k)\},k,1}(x,q)=1$,
(2) For $j=1,\dots,k-1$, $U^q_{\{(1k)\},k,kn+j}(x,q)=\Delta^q_{kn+j}(U^q_{\{(1k)\},k,kn+j-1}(x,q))$, and
(3) $U^q_{\{(1k)\},k,kn+k}(x,q)=\Gamma^q_{kn+k}(U^q_{\{(1k)\},k,kn+k-1}(x,q))$.

Then we can show that if $U^q_{\{(1k)\},k,kn+j}(x,q)=\sum_{s=0}^n U^{s,q}_{\{(1k)\},k,kn+j}x^s$, then we have the following.

Theorem 2.4. *For all $0\le j\le k-1$ and all n such that $kn+j>0$, we have*

$$U^{s,q}_{\{(1k)\},k,kn+j} =$$
$$[(k-1)n+j]_q! \sum_{r=0}^{s} (-1)^{s-r} q^{\binom{s}{2}-\binom{r}{2}-r(s-r)} \left[(k-1)n+j+r\right]_q^n \begin{bmatrix}(k-1)n+j+r\\ r\end{bmatrix}_q \begin{bmatrix}kn+j+1\\ s-r\end{bmatrix}_q$$

and

$$U^{s,q}_{\{(1k)\},k,kn+j} =$$
$$[(k-1)n+j]! \sum_{r=0}^{n-s} (-1)^{n-s-r} q^{\binom{n-s}{2}-\binom{r}{2}-r(n-s-r)-\binom{n+1}{2}+s(kn+j)} \left[1+r\right]_q^n \begin{bmatrix}(k-1)n+j+r\\ r\end{bmatrix}_q \begin{bmatrix}kn+j+1\\ n-s-r\end{bmatrix}_q$$

We can give a labeling procedure similar to one described for $Emaj$ to recursively define a statistic $(1k)maj$ such that $U^q_{\{(1k)\},k,kn+j}(x,q)=\sum_{\sigma\in S_{kn+j}} q^{(1k)maj(\sigma)} x^{asc_{\{(1k)\},k}(\sigma)}$.

It is interesting to note some differences between the proofs for the formulas of the above q-analogues and their respective $q=1$ case. In the $q=1$ case, we can give direct proofs of the extreme coefficients. That is, we can give direct combinatorial proofs of the fact that

$$U^0_{\{(1k)\},k,kn+j} = ((k-1)n+j)!\,((k-1)n+j)^n \text{ and}$$
$$U^n_{\{(1k)\},k,kn+j} = ((k-1)n+j)!$$

Then the two formulas for $U^s_{\{(1k)\},k,kn+j}$ arise by, first, starting with the formulas for $U^0_{\{(1k)\},k,kn+j}$ and using the recursion to iterate up to the formulas for $U^s_{\{(1k)\},k,kn+j}$ and, second, starting with the formulas for $U^n_{\{(1k)\},k,kn+j}$ and using the recursions to iterate down to the formulas for $U^{n-s}_{\{(1k)\},k,kn+j}$.

In the general q-case, the strategy does work since we can not give direct combinatorial proofs of both of the formulas for the coefficients $U^{0,q}_{\{(1k)\},k,kn+j}$ and $U^{n,q}_{\{(1k)\},k,kn+j}$ using our statistics $q^{(1k)maj(\sigma)}x^{asc_{\{(1k)\},k}(\sigma)}$. Thus we have to prove our formulas by assuming that they hold at level $kn+j$ and use the recursions to prove that they hold at level $kn+j+1$ for $j=0,\ldots,k-1$ and $n \geq 0$. In such a case, one has to be careful of what happens at the extreme cases, $U^{-1,q}_{\{(1k)\},k,kn+j}$ and $U^{n+1,q}_{\{(1k)\},k,kn+j}$ where the answers must be 0 by our definitions, but where our formulas still make sense. That is, typically, our formulas look like a sum from either 0 to s or from 0 to $n-s$. So, in one of the extreme cases, we can interpret the sum in our formula as the empty sum and is therefore zero by convention. In the other extreme case, we need to show explicitly that our formulas are zero when they are summing from 0 to $n+1$. It turns out that in such cases, we can give a direct combinatorial proof that the desired coefficient is 0. For example, we can give a direct combinatorial proof of the following.

THEOREM 2.5. *For all positive integers $k, n, j, z_1, \ldots, z_n$ and any function $\theta(r)$ where $kn+j > 0$, $0 < z_i < (k-1)n+j$, and $\theta(r+1) = \theta(r) - (n-r)$,*

$$\sum_{r=0}^{n+1} (-1)^{n+1-r} q^{\theta(r)} \prod_{i=1}^{n} [z_i + r]_q \begin{bmatrix} kn+j+1 \\ (k-1)n+j, r, n+1-r \end{bmatrix}_q = 0.$$

In [**L06**], Liese provided another formula for $U^{s}_{\{(1k)\},k,kn+j}$, which was obtained by directly applying inclusion-exclusion. That is, he showed that

$$U^{s}_{\{(1k)\},k,kn+j} = \sum_{r=s}^{n} (-1)^{r-s} \binom{r}{s} (kn+j-r)! S_{n+1,n+1-r}$$

where $S_{n,k}$ is the Stirling number of the second kind, i.e. $S_{n,k}$ is the number of set partitions of $\{1,\ldots,n\}$ into k parts. We conjecture a q-analogue of this formula as well.

$$U^{s,q}_{\{(1k)\},k,kn+j} = \sum_{r=s}^{n} (-1)^{r-s} q^{\binom{n+1-s}{2} + r((k-1)n+j) - \binom{n+1}{2} + ns} \begin{bmatrix} r \\ s \end{bmatrix}_q [kn+j-r]_q! S^q_{n+1,n+1-r}$$

where $S^q_{n,k}$ is the q-Stirling number of the second kind that defined by the following recursion.

$$S^q_{0,0} = 1, S^q_{n,k} = 0 \text{ if } k < 0 \text{ or } k > n, \text{ and,}$$
$$S^q_{n,k} = S^q_{n-1,k-1} + [k]_q S^q_{n-1,k} \text{ if } 0 \leq k \leq n.$$

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