LONGEST ALTERNATING SUBSEQUENCES IN PATTERN-RESTRICTED PERMUTATIONS

Ghassan Firro and Toufik Mansour

Department of Mathematics, University of Haifa, 31905 Haifa, Israel.

{gferro,toufik}@math.haifa.ac.il

ABSTRACT

We determine the limiting distributions of the longest alternating subsequence for pattern-restricted permutations in which the pattern is any one of the six patterns of length three. This work is inspired by results of Stanley concerning limiting distributions of the lengths of the longest alternating subsequences in random permutations, results of Widom on the limiting distribution of longest alternating subsequence in random permutations, and results of Deutsch, Hildebrand and Wilf on the limiting distribution of the longest increasing subsequences for pattern-restricted permutations,

2000 Mathematics Subject Classification: 05A05, 05A15, 05A16

Key words: limiting distributions, alternating sequences, functional equations, kernel methods, restricted permutations.

1. Introduction

Let S_n be the symmetric group of permutations of $1, 2, \dots, n$ and let $\pi = \pi_1\pi_2\dots\pi_n \in S_n$. An increasing subsequence in π of length ℓ is a subsequence $\pi_{i_1}\pi_{i_2}\dots\pi_{i_\ell}$ satisfying $\pi_{i_1} < \pi_{i_2} < \cdots < \pi_{i_\ell}$. Several authors have studied properties of the length of the longest increasing subsequence $is_n(\pi)$ of a permutation π (for an analogous theory for longest increasing subsequences in coloured permutations see [1]). Logan-Shepp [5] and Vershik-Kerov [8] have shown that the asymptotic expectation $E(n)$ of $is_n(\pi)$ satisfies $E(n) = \frac{1}{n!}\sum_{\pi\in S_n} is_n(\pi) \sim 2\sqrt{n}$ when $n \to \infty$. Recently, Stanley [7] (for the case of k-ary words see [6]) developed an analogous theory for alternating subsequences, *i.e.* subsequences $\pi_{i_1}\pi_{i_2}\dots\pi_{i_\ell}$ of π satisfying $\pi_{i_1} > \pi_{i_2} < \pi_{i_3} > \pi_{i_4} < \cdots \pi_{i_\ell}$. Stanley proved that the mean of the longest alternating subsequence $al_n(\pi)$ in a permutation $\pi \in S_n$ is $\frac{4n+1}{6}$ for $n \geq 2$, and the variance of $al_n(\pi)$ is $\frac{8}{45}n - \frac{13}{180}$ for $n \geq 4$. Also, Widom [9] showed the limiting distribution $\lim_{n\to\infty} Prob(al_n(\pi) \leq 2n/3 + t\sqrt{n})$ to be Gaussian with variance $\frac{8}{45}$.

Let $\pi \in S_n$ and $\tau \in S_k$ be two permutations. We say that π *avoids*τ, or is τ-*avoiding*, if there do not exist $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi_{i_1}\dots\pi_{i_k}$ is order-isomorphic to τ; in such a context, τ is usually called a *pattern*. The set of all τ-avoiding permutations in S_n is denoted $S_n(\tau)$. It is well known that the number of τ-avoiding permutations of length n, $\tau \in S_3$, is given by $c_n = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number (see [3, 4]).

Deutsch, Hildebrand and Wilf [2] generalized the above results to study the limiting distribution of the longest increasing subsequences in pattern-restricted permutations, for which the pattern is any one of the six patterns of length three. Inspired by the results of Stanley [7], and results of Deutsch, Hildebrand and Wilf [2], we study the limiting distribution of the longest alternating subsequences for pattern-restricted permutations in which the pattern is any one of the six patterns of length three. Our main results may be formulated as follows.

Theorem 1.1. *Let $\tau \in S_3$. In the class of τ-avoiding permutations of length n, the length of the longest alternating subsequence has mean μ_τ with*

τ	123	132, 231, 321	213, 312
μ_τ	$\frac{2n^2-5n-9}{2(2n-1)}$	$\frac{(n-1)(2n+5)}{2(2n-1)}$	$\frac{n+1}{2}$

and variance σ_τ with

τ	123	132, 231, 321	213, 312
σ_τ	$\frac{(n+1)(8n^3-50n^2+101n+9)}{4(2n-1)^2(2n-3)}$	$\frac{(n+1)(8n^3-42n^2+73n-15)}{4(2n-1)^2(2n-3)}$	$\frac{(n+1)(4n^2-15n+15)}{4(2n-1)(2n-3)}$

Moreover, the (almost normalized) random variable $X_n^\tau(\pi) = \frac{al_n(\pi)-\frac{n}{2}}{\frac{1}{2}\sqrt{n}}$, defined for all permutations of length n that avoid the pattern τ, satisfies

$$\lim_{n\to\infty} \sqrt{n} \cdot Prob(X_n^\tau(\pi) = \theta) = \frac{4}{\sqrt{\pi}} e^{-\theta^2}.$$

In addition, we find an explicit formula for the number of τ-avoiding permutations π of length n satisfying $al_n(\pi) = m$, where τ is any pattern in S_3.

Theorem 1.2. *The number of 123-avoiding permutations π of length $n \geq 3$ satisfying $al_n(\pi) = m$, with $1 \leq m \leq n$, is given by*

$$\sum_{j=\lfloor\frac{m+2}{2}\rfloor}^{n+1} \frac{(-1)^{n-j-\lfloor\frac{m-1}{2}\rfloor}}{j+1}\binom{2j}{j}\binom{j-2}{\lfloor\frac{m-2}{2}\rfloor}\binom{j-\lfloor\frac{m+1}{2}\rfloor}{n-j-\lfloor\frac{m-1}{2}\rfloor}.$$

The number of 321-avoiding (132-avoiding, 231-avoiding) permutations π of length $n \geq 1$ satisfying $al_n(\pi) = m$, with $1 \leq m \leq n$, is given by

$$\sum_{j=\lfloor\frac{m+1}{2}\rfloor}^{n+1} \frac{(-1)^{n-j-\lfloor\frac{m}{2}\rfloor}}{j+1}\binom{2j}{j}\binom{j-1}{\lfloor\frac{m-1}{2}\rfloor}\binom{j-\lfloor\frac{m}{2}\rfloor}{n-j-\lfloor\frac{m}{2}\rfloor}.$$

The number of 312-avoiding (213-avoiding) permutations π of length n satisfying $al_n(\pi) = m$, with $1 \leq m \leq n$, is given by

$$\sum_{j=\lfloor\frac{m-1}{2}\rfloor}^{n+1} \frac{(-1)^{n-j-\lfloor\frac{m+1}{2}\rfloor}}{j+1}\binom{2j}{j}\binom{j}{\lfloor\frac{m}{2}\rfloor}\binom{j-\lfloor\frac{m-1}{2}\rfloor}{n-j-\lfloor\frac{m+1}{2}\rfloor}.$$

Our method gives the following asymptotics.

Corollary 1.3. *Fix $m > 0$ and let $n \to \infty$. Then*

- *the number of 123-avoiding permutations π of length n satisfying $al_n(\pi) = m$ is asymptotically $\frac{n^{2\lfloor\frac{m}{2}\rfloor-4}}{(\lfloor\frac{m}{2}-1\rfloor)!(\lfloor\frac{m}{2}\rfloor-2)!} \cdot 2^{n-2\lfloor\frac{m}{2}\rfloor+5}$.*

- *the number of* 321*-avoiding (*132*-avoiding,* 231*-avoiding) permutations* π *of length* n *satisfying* $al_n(\pi) = m$ *is asymptotically* $\frac{n^{2\lfloor\frac{m+1}{2}\rfloor-4}}{(\lfloor\frac{m-1}{2}\rfloor)!(\lfloor\frac{m-3}{2}\rfloor)!}\cdot 2^{n-2\lfloor\frac{m+1}{2}\rfloor+4}$.
- *the number of* 312*-avoiding (*213*-avoiding) permutations* π *of length* n *satisfying* $al_n(\pi) = m$ *is asymptotically given by* $\frac{n^{2\lfloor\frac{m}{2}\rfloor-2}}{(\lfloor\frac{m}{2}\rfloor)!(\lfloor\frac{m}{2}\rfloor-1)!}\cdot 2^{n-2\lfloor\frac{m}{2}\rfloor+1}$.

References

[1] A. Borodin, Longest increasing subsequences of random colored permutations, *Electron. J. Combin.* **6** (1999) #R13.
[2] E. Deutsch, A.J. Hildebrand, and H.S. Wilf , Longest increasing subsequences in pattern-restricted permutations, *Elect. J. Combin.* **9:2** (2002) #R12.
[3] D. E. Knuth, The art of computer programming, Volume 1, Addison-Wesley, 3rd ed., 1997.
[4] D. E. Knuth, The art of computer programming, Volume 3, Addison-Wesley, 1973.
[5] B. F. Logan and L. A. Shepp, A variational problem for random Young tableaux, *Adv. Math.* **26** (1977) 206—222.
[6] T. Mansour, Longest alternating subsequences of k-ary words, Preprint.
[7] R. Stanley, Longest alternating subsequences of permutations, preprint, arXiv/0511419v1.
[8] A. M. Vershik and K. V. Kerov, Asymptotic behavior of the Plancherel measure of the symmetric group and the limit form of Young tableaux (Russian), *Dokl. Akad. Nauk SSSR* **223** (1977) 1024—1027. English translation: *Soviet Math. Dokl.* **233** (1977) 527—531.
[9] H. Widom, On the limiting distribution for the longest alternating sequence in a randon permutation, *Elect. J. Combin.* **13:1** (2006) #R25.