Counting Pattern-Avoiding Permutations with Perron and Frobenius

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This talk reports on joint work with Richard Ehrenborg and Sergey Kitaev, and gives a new method for counting consecutive pattern-avoiding permutations using the spectral theory of integral operators on the unit $(k-1)$-cube. Let $\mathfrak{S}_n$ denote the symmetric group on n elements; a *pattern* of length k is subset S of $\mathfrak{S}_k$. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with no two coordinates equal, denote by $\Pi(x)$ the unique permutation in $\mathfrak{S}_n$ with $\pi_i < \pi_j$ if and only if $x_i < x_j$. A permutation $\sigma \in \mathfrak{S}_n$, $n \geq k$, avoids the pattern S if $\Pi(\sigma) \notin S$. In other words, no consecutive pattern in S occurs in the permutation σ.

Denote by $\alpha_n(S)$ the number of permutations in $\mathfrak{S}_n$ that avoid the pattern S, so that the ratio

$$\frac{\alpha_n(S)}{n!} \tag{1}$$

gives the probability that a given permutation in $\mathfrak{S}_n$ avoids the pattern S. We wish to compute the asymptotics of $\alpha_n(S)$ as $n \to \infty$.

We will show how to associate a positivity-preserving integral operator T_S to a pattern S so that the asymptotics of the ratio (1) are determined by the spectrum of T_S. Kreĭn and Rutman's generalization of the classical Perron-Frobenius theorem allows one to conclude that

$$\frac{\alpha_n(S)}{n!} \sim \rho(S)^n$$

as $n \to \infty$, where $\rho(S)$ is the modulus of the largest eigenvalue of T_S. Moreover, in case the spectrum of T_S is known, one can obtain a complete asymptotic expansion. We illustrate the method by computing asymptotics of the ratio (1) for $S = \{123\}$, $S = \{213\}$, and $S = \{123, 321\}$. We also establish a number of more general theorems relating the spectral theory of T_S to the enumeration problem at hand. Finally, we give a simple explicit formula relating our integral operator approach to the generating function approach used, for example, by Elizalde and Noy, some of whose results we have generalized.