Determining Lower Bounds for Packing Densities of Non-layered Patterns Using Weighted Templates

Cathleen Battiste Presutti
Bryn Mawr College

In Albert, Atkinson, Handley, Holton & Stromquist [1], the lower bound for the packing density of the pattern 2413, δ(2413), was determined by starting with the permutation 35927146, which contains a relatively large number of occurrences (17) of 2413-pattern and restricting consideration to permutations of the form $\sigma = \sigma_3\sigma_5\sigma_8\sigma_2\sigma_7\sigma_1\sigma_4\sigma_6$ where $\sigma_i < \sigma_{i+1}$ and $|\sigma_i| = |\sigma_{i+1}|$ for each i, and where each σ_i is recursively composed in this same fashion. Furthermore, if $n = |\sigma|$, then $\frac{n}{8} = |\sigma_i|$ for each i. The probability p_n of obtaining a 2413 pattern in σ by considering two different scenarios can be determined. The first scenario is that all four points are picked from the same σ_i. The probability of this occurring is the product of the number of σ_i's, the probability of picking all four points in the same σ_i, and the probability $p_{\frac{n}{8}}$ that the four points formed a 2413 pattern in σ_i, which equals

$$8\left(\frac{1}{8}\right)^4 p_{\frac{n}{8}} = \frac{1}{512} p_{\frac{n}{8}}.$$

The second situation is that each point is picked from a distinct σ_i. The probability of this occurring is the product of the number of occurrences of 2413 in 35827146, the number of different orderings of four elements, and the probability of picking one point from each σ_i, which equals

$$17 \cdot 4! \cdot \left(\frac{1}{8}\right)^4 = \frac{51}{512}.$$

Therefore $p_n = \frac{1}{512} p_{\frac{n}{8}} + \frac{51}{512}$.

By taking the limit as n approaches infinity, we know that p_n approaches a limit p which leads us to

$$p = \frac{1}{512} p + \frac{51}{512} \text{ and } p = \frac{51}{511}.$$

Since $p_n \le \delta_n(2413)$, it is clear that $p \le \delta(2413)$ and finally $\frac{51}{511} \le \delta(2413)$.

This method can be formalized by calling the permutation 35927146 a "template". The construction described above can be applied to any pattern and any template, and it gives a lower bound for the packing density of a pattern.

In this paper, we describe a generalization of this method involving *weighted* templates. In general, improved lower bounds for the packing density of a non-layered pattern π of size n can be found using the following template. We start with a permutation σ_{mn} that has mn slots, i.e. a multiple of n, and that contains an acceptable number of occurrences of the given pattern π. Assign to each slot s_i, $1 \le i \le mn$, in σ a variables x_j where $1 \le j \le m$. The x_j's will represent the weight of the slot, or the probability of that slot being part of an occurrence of π in σ. Using these slot assignments, we create a polynomial that computes the probability that an occurrence of π in σ uses four distinct slots. From this, we construct a polynomial function $f(x_1, x_2, ..., x_m)$ and introduce some relational constraints on the variables, $x_1, x_2, ..., x_m$, so that our function reduces to $f(x_k)$ for some $1 \le k \le m$. In doing so, we can use simple calculus to determine if there exists a critical point w_k that produces a local maximum and that satisfies the relational constraints. If one exists, this critical point w_k is the weight for slots corresponding to x_k and then is used to find the weights w_j, $1 \le j \le m$, of the remaining slots.

Now we return to the recursive probability equation from AAHHS. Using the weighted probabilities yields

$$p_{mn} = \left(\sum_{j=1}^{m} a_j w_j^m \right) p_m + 4! \cdot f(w_1, w_2, ..., w_m), \text{ where } a_j \text{ is the multiplicity of } w_j .$$

By taking the limit as n approaches infinity, we know that p_{mn} approaches a limit p which leads us to

$$p = \left(\sum_{j=1}^{m} a_j w_j^m \right) p + 4! \cdot f(w_1, w_2, ..., w_m)$$

Solving for p provides us with the desired lower bound for the packing density of the pattern π, $\delta(\pi)$.

In this paper, we describe applications of the weighted template method to packings of 2413, using templates of size 8, 12, and 16 with optimized weights. The best result gives a lower bound of 0.10543470687778887 for the packing density of 2413, which is an improvement on the result from AAHHS.

It is important to note that in 2003 Michael Albert, Nik Ruskuc, and Imre Leader were able to do better than the lower bound of 51/511. Their packing permutations also can be used as templates for getting improved lower bounds for the packing density of 2413. Experiments done by Vince Vatter using simulated annealing also suggested that permutation structures examined by Albert, Ruskuc, and Leader were strong candidates.

References

[1] M. H. Albert, M. D. Atkinson, C. C. Handley, D. A. Holton, W. Stromquist: On packing densities of permutations, *Electronic Journal of Combinatorics* **9** (2002), #R5