A Bijection on $\{123, 132\}$-avoiding Multiset Permutations

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Recall that two finite sequences a and b are *order isomorphic* if a and b have the same length and $a_i \leq a_j \iff b_i \leq b_j$ for all i and j. If $\sigma \in [k]^n$ is a string of n letters over the alphabet $[k] = \{1, \ldots, k\}$ and $\tau \in [l]^m$ is a map from $[m]$ onto $[l]$ (i.e. τ contains all letters from 1 to l), then we say that σ *contains* the pattern τ if σ has a subsequence order isomorphic to τ. If σ does not contain τ, then we say that σ *avoids* τ.

In his thesis [1], Burstein used analytic methods to find the generating function for multiset permutations on an alphabet of size k which simultaneously avoid the patterns 123 and 132. Last year [2], Mansour gave a beautiful bijective proof that $\{132, 123\}$-avoiding words and $\{213, 231\}$-avoiding words on $[k]^n$ are equinumerous. In this paper, I give a purely bijective proof of these results. First, I detail a bijection between $\{132, 123\}$-avoiding permutations, and 2-colored decreasing sequences. I believe there is also a bijection between $\{231, 213\}$-avoiding words and 2-colored increasing sequences. The composition of these two maps yields a new bijection between the two sets of restricted words.

Theorem. *[1] Let $S_k(n)$ be the number of words $\sigma \in [k]^n$ $(k \geq 0)$ which simultaneously avoid the patterns* 123 *and* 132*, and let $F_k(x) = \sum S_k(n)x^n$. Then*

$$F_k(x) = \frac{1}{2(1-x)(1-2x)^{k-1}} + \frac{1}{2(1-x)}$$

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First, notice that for $k \leq 2$, *all* words on $[k]^n$ avoid 123 and 132, so $F_0(x) = 1$, $F_1(x) = \frac{1}{1-x}$, and $F_2(x) = \frac{1}{1-2x}$, which are all of the desired form. Now, we turn our attention to general k, and consider the following proposition.

Proposition. *The number of $\{123, 132\}$-avoiding words on $[k]^n$ with at least one k is $\sum_{i=0}^{n-1} \binom{k-2+i}{k-2} 2^i$.*

Proof. We prove the proposition with a bijection.

By the standard "stars and bars" computation, the number of decreasing sequences of length i on the alphabet $[k-1]$ is $\binom{k-2+i}{k-2}$. Thus, $\sum_{i=0}^{n-1}\binom{k-2+i}{k-2}2^i$ counts the number of 2-colored decreasing sequences on $[k-1]$ of length at most $n-1$.

Now we exhibit a bijection from the set of 2-colored decreasing sequences on $[k-1]$ of length at most $n-1$ to $\{123, 132\}$-avoiding words on $[k]^n$ with at least one k.

Algorithm 1.
Input: $s = (s_1, ..., s_m)$, a 2-colored decreasing sequence on $[k-1]$ of length m $(< n)$.
Output: $w \in [k]^n$, a $\{123, 132\}$-avoiding word with at least one k.

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max:=k
w:=k

for i from 1 to m do
    l:=length(w)
    if color(s_i)=R then
      w:=w_1, ..., w_l, s_i
    if color(s_i)=L, s_{i-1} > s_i, and color(s_{i-1})=R then
      w:=w_1, ..., w_{l-1}, s_i, w_l
      max:=s_{i-1}
    if color(s_i)=L and there does not exist s_j = s_i with j < i
     and color(s_j)=R then
      w:=w_1, ..., w_{l-1}, s_i, w_l
    if color(s_i)=L and there exists s_j = s_i with j < i and color(s_j)=R then
      w:=w_1, ..., w_{l-1}, max, w_l

If length(w) < n then
    w := w, max^{n-l}.

Return w.
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Example 1. In general, the 2-coloring of the sequence s instructs us whether to insert s_i to the right (R) or the left (L) of the last letter of w with a little added bookkeeping.
Let $k = 3$ and $n = 5$.
input: (2(R),1(L),1(R),1(L))
First, initialize $w = 3$, $max := 3$.
Since $s_1 = 2$ and color(s_1) = R, $w = 32$, $max := 3$.
Since $s_2 = 1$ and color(s_2) = L, $w = 312$, $max := 2$.
Since $s_3 = 1$ and color(s_3) = R, $w = 3121$, $max := 2$.
Since $s_4 = 1$ and color(s_4) = L, $w = 31221$, $max := 2$.
Since all entries of s have been exhausted and w has length n, w is the desired

$\{123, 132\}$-avoiding word on $[3]^5$ with at least one 3.

To see that the algorithm provides a bijection, it is enough to see that it has an inverse. Appealing to the general intuition that Algorithm 1 places an R-colored (resp. L-colored) entry of a decreasing sequence to the right (resp. left) of the last entry of the word, an inverse map should reverse this process. All letters to the left of the first k are colored with L. After this, the inverse map finds the largest possible uncolored entry to add to the decreasing sequence and labels it R, labelling everything smaller and to the left of this entry with an L. Consider the following:

Algorithm 2.
Input: $w \in [k]^n$, a $\{123, 132\}$-avoiding word with at least one k.
Output: D, a decreasing sequence on $[k-1]$ of length m ($< n$) and C, a sequence of colors of the same length.

```
C:=the empty word
D:=the empty word
i:=1

while w_i ≠ k
    D:=D,w_i,     C:=C,L,    i:=i+1

while there are uncolored entries of w
    Let m:=min(max(uncolored entries of w),last entry of D, k-1).
    Let curr be the position of the first uncolored occurrence of m.
    D:=D, w_curr,     C:=C, R

    Let w_{j_1},...,w_{j_c} be the uncolored elements of w to the left of curr
    (other than the first occurrence of k)
      for i from 1 to c
        If w_{j_i} ≤ w_curr then
          D:=D, w_j,     C:=C, L
        If w_{j_i} > w_curr then
          D:=D, w_curr,     C:=C, L

If all elements of w have been colored, or all remaining entries of w
are larger than the last entry of D, then return [D,C].
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Example 2. Let $k = 3$ and $w = 31221$.
There are no entries to the left of the first 3, so $m := \min\{\max\{1, 2, 2, 1\}, 2\} = 2$.
w_3 is the first uncolored occurence of m, and w_2 is the only uncolored element to the left of w_3. Thus, D=[2,1], C=[R,L].
Now, $m := \min\{\max\{2, 1\}, 1\} = 1$.

w_5 is the first uncolored occurence of m and w_4 is the only uncolored element to the left of w_5. Thus, D=[2,1,1,1] C=[R,L,R,L].

We have exhibited an inverse map for Algorithm 1. Now, we check that Algorithm 1 always maps 2-colored decreasing sequences to $\{123, 132\}$-avoiding words with at least one k. It is clear from the algorithm that w contains at least one k.

To see that w is 132-avoiding, recall that s is a decreasing sequence and that the algorithm inserts each entry of s into w as either the last, or next to last entry of w. Assume that s_i and s_{i+1} form the 32 part of a 132 pattern. Then, some entry s_k, $k < i$ must play the role of 1 in the 132 pattern, which contradicts the fact that s is decreasing. Otherwise, s_{i+1} plays the role of the 1 in the 132 pattern, which implies that s_{i+1} was inserted earlier than the next to last element of w, a contradiction.

To see that w is 123-avoiding, notice that the only way for two entries of s to be inserted into w in increasing order is if $s_i > s_{i+1}$, and color(s_{i+1}) = L, color(s_i) = R. However, in this case, the algorithm reassigns $max := s_i$, and no entry larger than max is inserted into w during the rest of the algorithm. Thus it is impossible to create a 123 pattern.

Now, it is clear that the algorithms provide a bijection between the set of 2-colored decreasing sequences on $[k-1]$ of length at most $n-1$ and $\{123, 132\}$-avoiding words on $[k]^n$ with at least one k.

□

With this proposition, we are in a position to prove the following lemma, which, with induction, proves the theorem.

Lemma. *[2] Let $W_k(n)$ be the number of words $\sigma \in [k]^n$ which simultaneously avoid the patterns 123 and 132 and contain at least one k, and let $G_k(x) = \frac{x}{(1-x)(1-2x)^{k-1}}$. Then, $\sum W_k(n)x^n = F_k(x) - F_{k-1}(x) = G_k(x)$.*

Proof.

$$G_k(x) = \frac{x}{(1-x)(1-2x)^{k-1}} = \frac{x}{1-x}\frac{1}{(1-2x)^{k-1}} = \frac{x}{1-x}\sum_{n\geq 0}\binom{k-2+n}{k-2}2^n x^n$$

The coefficient of x^n in $G_k(x)$ is $\sum_{i=0}^{n-1}\binom{k-2+i}{k-2}2^i$. By the proposition, this is the number of $\{123, 132\}$-avoiding words on $[k]^n$ with at least one k, so we are done. □

A similar bijection between $\{213, 231\}$-avoiding words on $[k]^n$ and 2-colored sequences is in progress.

References

[1] Alexander Burstein, *Enumeration of Words with Forbidden Patterns*, Ph.D. thesis, University of Pennsylvania, 1998.

[2] Toufik Mansour, *Restricted 132-avoiding k-ary words, Chebyshev polynomials, and Continued fractions.*, The Third International Conference on Permutation Patterns, University of Florida, Gainesville, Florida, March 7–11, 2005.