Permutations and words counted by consecutive patterns

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Abstract. Generating functions which count occurrences of consecutive sequences in a permutation or a word which matches a given pattern are studied by exploiting the combinatorics associated with symmetric functions. Our theorems take the generating function for the number of permutations which do not contain a certain pattern and give generating functions refining permutations by the both the total number of pattern matches and the number of nonoverlapping pattern matches. Our methods allow us to give new proofs of several previously recorded results on this topic as well as to prove new extensions and new q-analogues of such results.

1. Introduction and Preliminaries

Given a sequence $\sigma = \sigma_1 \cdots \sigma_n$ of distinct integers, let $\mathrm{red}(\sigma)$ be the permutation found by replacing the i^{th} largest integer that appears in σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $\mathrm{red}(\sigma) = 1\ 4\ 3\ 2$. Given a permutation τ in the symmetric group S_j, define a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ to have a τ-match at place i provided $\mathrm{red}(\sigma_i \cdots \sigma_{i+j-1}) = \tau$. Let $\tau\text{-mch}(\sigma)$ be the number of τ-matches in the permutation σ.

To prevent confusion, we note that a permutation not having a τ-match different than a permutation being τ-avoiding. A permutation is called τ-avoiding if there are no indices $i_1 < \cdots < i_j$ such that $\mathrm{red}(\sigma_{i_1} \cdots \sigma_{i_j}) = \tau$. For example, if $\tau = 2\ 1\ 4\ 3$, then the permutation $3\ 2\ 1\ 4\ 6\ 5$ does not have a τ-match but it does not avoid τ since $\mathrm{red}(2\ 1\ 6\ 5) = \tau$.

Recent publications have analyzed the distribution of τ-matches in permutations and several nice theorems have been proved [**EN03, Kit03, Kit**]. One specific result is as follows. Let $\tau\text{-nlap}(\sigma)$ be the maximum number of nonoverlapping τ-matches in σ where two τ-matches are said to overlap if they contain any of the same integers. It was published both in a paper and the doctoral dissertation by Sergey Kitaev [**Kit03, Kit**] that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\sigma)} = \frac{A(t)}{(1-x) + x(1-t)A(t)} \tag{1.1}$$

where $A(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}|$. In other words, if the exponential generating function for the number of permutations in S_n without any τ-matches is known, then so is the exponential generating function for the entire distribution of the statistic τ-nlap.

As a starting point for our present work, (1.1) will be proved in a new way by exploiting the relationship between the elementary and homogeneous symmetric functions. The class of permutation patterns for which Kitaev's theorem holds will be enlarged in the process.

In particular, suppose $\Upsilon \subseteq S_j$. We say that a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has an Υ-match at place i provided $\mathrm{red}(\sigma_i \cdots \sigma_{i+j-1}) \in \Upsilon$. Let $\Upsilon\text{-mch}(\sigma)$ and $\Upsilon\text{-nlap}(\sigma)$ be the number of Υ-matches and nonoverlapping Υ matches in σ, respectively. We will prove an analogue of (1.1) where every τ in (1.1) is replaced with an Υ.

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It may be difficult to find the exponential generating function counting the number of permutations in S_n with τ-mch$(\sigma) = 0$ and thus finding the generating function $A(t)$ in the statement of Kitaev's theorem may be difficult. Therefore, we will develop an alternate method of understanding the expression on the right hand side of (1.1).

After this is done, we will show how slight modifications to our new proof can yield results about inversion counts of permutations refined by the maximum number of nonoverlapping pattern matches. This will provide a q-analogue for (1.1). Using another variant on our proof, we will provide results about pattern matches for m-tuples of permutations. All of these ideas will be included in Section 2.

Throughout this paper we will use the relationships between symmetric functions as a tool to gather facts about the symmetric group. The idea of extracting information about permutation statistics through symmetric function theory has been used for decades, but the methods of this paper—defining a homomorphism mapping the elementary symmetric functions to a polynomial ring—was first given by Francesco Brenti [**Bre90, Bre93**]. Desiree Beck and Jeff Remmel gave completely combinatorial proofs of Brenti's results which allowed them to give q-analogues of those results as well as to prove analogues of those results for other groups [**Bec93, BR95**]. Later, Jeff Remmel and the current author described how to understand a well known formula of Garsia and Gessel with this view [**MR**]. It is this approach which is closest to our own.

Let $\Lambda = \oplus_{n\geq 0}\Lambda_n$ denote the ring of symmetric functions where Λ_n is the space of homogeneous symmetric functions of degree n. The elementary symmetric function e_n is defined by $\sum_{n\geq 0} e_n t^n = (1+x_1t)(1+x_2t)\cdots$. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an integer partition; that is, λ is a finite sequence of weakly increasing nonnegative integers. We will let $\ell(\lambda)$ denote the number of nonzero integers in λ. If the sum of these integers is n, we say that λ is a partition of n and write $\lambda \vdash n$. For any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $e_\lambda = e_{\lambda_1}\cdots e_{\lambda_\ell}$. It is well known that that $\{e_\lambda : \lambda \text{ is a partition of } n\}$ is a basis for Λ_n and hence $\{e_n : n \geq 0\}$ generate Λ as a ring.

The homogeneous symmetric function h_n is defined by $\sum_{n\geq 0} h_n t^n = \frac{1}{1-x_1t}\cdot\frac{1}{1-x_2t}\cdots$. Therefore,

$$\sum_{n\geq 0} h_n t^n = \frac{1}{1-x_1t}\cdot\frac{1}{1-x_2t}\cdots = ((1-x_1t)(1-x_2t)\cdots)^{-1} = \left(\sum_{n\geq 0} e_n(-t)^n\right)^{-1}. \tag{1.2}$$

We shall see that we can prove Kitaev's Theorem and its generalization by applying a ring homomorphism ξ to both sides of 1.2.

2. Nonoverlapping Consecutive Patterns in Permutations

With an eye toward proving Kitaev's Theorem, we now define a few auxiliary sets associated with a given permutation $\tau \in S_j$. For a permutation $\sigma \in S_n$, let

$$\text{Mch}_\tau(\sigma) = \{i : \text{red}(\sigma_{i+1}\cdots\sigma_{i+j}) = \tau\} \qquad \text{and}$$
$$I_\tau = \{1 \leq i < j : \text{there exist } \sigma \in S_{j+i} \text{ such that } \text{Mch}_\tau(\sigma) = \{0, i\}\}.$$

One (or one's computer) can find every element in the set I_τ for any $\tau \in S_j$ by finding the set $\text{Mch}_\tau(\sigma)$ for all $\sigma \in S_{j+i}$ for $i = 1, \ldots, j-1$.

Let I_τ^* be the set of all words with letters in the set I_τ. We let ϵ denote the empty word. If $w = w_1\cdots w_n \in I_\tau^*$ is word with n-letters, we define

$$\ell(w) = n, \qquad \sum w = \sum_{i=1}^{n} w_i, \qquad \text{and} \qquad ||w|| = j + \sum w.$$

In the special case where $w = \epsilon$, we let $\ell(w) = 0$ and $\sum w = 0$. Let

$$A_\tau = \{w \in I_\tau^* : \ell(w) \geq 2 \text{ and } \sum w < j\} \qquad \text{and}$$
$$B_{u,\tau} = \{w_1\cdots w_n \in I_\tau^* : \sum w_2\cdots w_n + \sum u < j \leq \sum w_1\cdots w_n + \sum u\}$$

for each word $u \in I_\tau^*$ with $\sum u < j$.

To help us understand these sets, let us look at the situation for four different permutations τ. These examples should provide insight into the kinds of situations which may arise.

First, consider $\alpha = 2\ 1\ 4\ 3$. We have that $\mathrm{Mch}_\alpha(2\ 1\ 4\ 3\ 6\ 5) = \{0, 2\}$ and $\mathrm{Mch}_\alpha(3\ 2\ 5\ 4\ 1\ 6\ 7) = \{0, 3\}$. Therefore $2, 3 \in I_\alpha$. A quick check can show that $1 \notin I_\alpha$; giving that $I_\alpha = \{2, 3\}$. The set I_α^* is the set of all words in the letters 2 and 3. Therefore $A_\alpha = \varnothing$ since every word w of length ≥ 2 in the letters 2 and 3 has $\sum w \geq 4$. The sets $B_{2,\alpha} = B_{3,\alpha}$ are both equal to $\{2, 3\}$ in this case.

Next, consider $\beta_j = j \cdots 2\ 1$. It may be shown that $I_{\beta_j} = \{1\}$, and therefore $A_{\beta_j} = \{1^k : 2 \leq k < j\}$ (here, 1^k denotes the word $1 \cdots 1$). If $1 \leq i < j$ then $B_{i,\beta_j} = \{1^{j-i}\}$.

When $\gamma = 2\ 1\ 4\ 3\ 6\ 5$, $I_\gamma = \{2, 5\}$, $A_\gamma = \{22\}$, $B_{2,\gamma} = B_{5,\gamma} = \{2, 5\}$ and $B_{22,\gamma} = \{22, 5\}$.

As our last example, let $\delta = 1\ 5\ 2\ 7\ 3\ 8\ 4\ 9\ 6$. Via a computer search, I_δ has been shown to equal $\{2, 4, 6, 8\}$. We are grateful to Jeff Liese; he found δ and I_δ. In this situation,

$$\begin{aligned}
A_\delta &= \{22, 24, 26, 42, 44, 62, 222, 224, 242, 422, 2222\},\\
B_{2,\delta} &= \{8, 26, 44, 62, 224, 242, 422, 2222\},\\
B_{4,\delta} &= B_{22,\delta} = \{6, 8, 24, 62, 42, 44, 222, 422\},\\
B_{6,\delta} &= B_{24,\delta} = B_{42,\delta} = B_{222,\delta} = \{4, 6, 8, 22\}, \qquad \text{and}\\
B_{8,\delta} &= B_{26,\delta} = B_{44,\delta} = B_{62,\delta} = B_{224,\delta} = B_{242,\delta} = B_{422,\delta} = \{2, 4, 6, 8\}.
\end{aligned}$$

Form a new alphabet

$$K_\tau = \{\overline{u} : u \in I_\tau\} \cup \{\overline{w} : w \in A_\tau\}.$$

We let $\Psi : K_\tau^* \to I_\tau^*$ be the function such that $\Psi(\epsilon) = \epsilon$ and $\Psi(\overline{w_1} \cdots \overline{w_n}) = w_1 \ldots w_n$. For example, if $\tau = \gamma = 2\ 1\ 4\ 3\ 5\ 6$ as above, then $\Psi(\overline{5}\ \overline{22}\ \overline{2}\ \overline{2}\ \overline{5}) = 522225$.

Define $\overline{J}_\tau$ in the following manner.

(1) $\epsilon \in \overline{J}_\tau$.
(2) $\overline{v} \in \overline{J}_\tau$ for all $v \in I_\tau$.
(3) If $\overline{w_1} \cdots \overline{w_n} \in \overline{J}_\tau$, then $\overline{u}\ \overline{w_1} \cdots \overline{w_n} \in \overline{J}_\tau$ for all $u \in B_{w_1,\tau}$.
(4) The only words in $\overline{J}_\tau$ are the result of applying one of the above rules.

Take $J_\tau = \Psi(\overline{J}_\tau)$.

As an example, consider taking $\tau = \gamma = 2\ 1\ 4\ 3\ 5\ 6$. Then

$$\overline{J}_\gamma = \{\epsilon, \overline{2}, \overline{5}, \overline{22}\ \overline{2}, \overline{5}\ \overline{2}, \overline{2}\ \overline{5}, \overline{5}\ \overline{5}, \overline{2}\ \overline{22}\ \overline{2}, \overline{5}\ \overline{22}\ \overline{2}, \overline{2}\ \overline{5}\ \overline{2}, \overline{5}\ \overline{5}\ \overline{222}\ \overline{2}\ \overline{5}, \overline{5}\ \overline{2}\ \overline{5}, \overline{2}\ \overline{5}\ \overline{5}, \overline{5}\ \overline{5}\ \overline{5}, \ldots\}$$

The final set we would like to define is as follows. Let

$$\mathcal{P}_w^\tau = \{\sigma \in S_{||w||} : \mathrm{Mch}_\tau(\sigma) = \{0, w_1, w_1 + w_2, \ldots, w_1 + w_2 + \cdots + w_n\}\}$$

and $\mathcal{P}_\epsilon^\tau = \{\tau\}$.

In order to find the exponential generating function for the number of permutations in S_n refined by the maximum number of nonoverlapping τ-matches, we will introduce a homomorphism on the ring of symmetric functions by defining it on e_n for $n \geq 0$. Let $f(n) = (-1)^n$ if $n = 0, 1$ and

$$f(n) = (1 - x) \sum_{\omega \in J_\tau, ||\omega|| = n} (-1)^{\overline{\ell}(\omega)} |\mathcal{P}_\omega^\tau|$$

otherwise. Let ξ be the ring homomorphism on Λ with the property that

$$\xi(e_n) = \frac{(-1)^n}{n!} f(n).$$

Then by using the combinatorics of the expansion of h_n in terms of the elementary symmetric functions to interpret $n!\xi(h_n)$ as a set of signed combinatorial objects and defining an appropriate sign reversing involution, we can prove the following.

Theorem 2.1. *For $\tau \in S_j$,*

$$n!\xi(h_n) = \sum_{T \in \mathcal{T}_\tau} w(T) = \sum_{T \in \mathcal{T}_\tau, \mathcal{I}_\tau(T) = T} w(T).$$

At this point, we can apply ξ to both sides of the identity in (1.2) to obtain

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in S_n}x^{\tau\text{-}nlap(\sigma)} = \frac{1}{1+\sum_{n=1}^{\infty}\xi(e_n)} = \frac{1}{1-t+\sum_{n=2}^{\infty}\frac{t^n}{n!}(1-x)\sum_{w\in J_\tau,||w||=n}(-1)^{\overline{\ell}(w)}|\mathcal{P}^{\tau}_{w}|}. \tag{2.1}$$

Then setting $x=0$ in the above equations,

$$A(t) = 1+\sum_{n=1}^{\infty}\frac{t^n}{n!}|\{\sigma\in S_n : \tau\text{-mch}(\sigma)=0\}| = \frac{1}{1-t+\sum_{n=2}^{\infty}\frac{t^n}{n!}\sum_{w\in J_\tau,||w||=n}(-1)^{\overline{\ell}(w)}|\mathcal{P}^{\tau}_{w}|}.$$

Thus

$$\sum_{n=2}^{\infty}\frac{t^n}{n!}\sum_{w\in J_\tau,||w||=n}(-1)^{\overline{\ell}(w)}|\mathcal{P}^{\tau}_{w}| = \frac{1}{A(t)}-(1-t). \tag{2.2}$$

Combining (2.1) and (2.2), we see that

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in S_n}x^{\tau\text{-nlap}(\sigma)} = \frac{1}{1+\sum_{n=1}^{\infty}\xi(e_n)} = \frac{1}{1-t+(1-x)(\frac{1}{A(t)}-(1-t))} = \frac{A(t)}{(1-x)-x(1-t)A(t)}.$$

Thus we get a new proof of Kitaev's theorem. Note that our methods gives the following corollary.

COROLLARY 2.2. *For any permutation τ,*

$$A(t) = 1+\sum_{n=1}^{\infty}\frac{t^n}{n!}|\{\sigma\in S_n : \tau\text{-mch}(\sigma)=0\}| = \frac{1}{1-t+\sum_{n=2}^{\infty}\frac{t^n}{n!}\sum_{w\in J_\tau,||w||=n}(-1)^{\overline{\ell}(w)}|\mathcal{P}^{\tau}_{w}|}.$$

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in S_n}x^{\tau\text{-nlap}(\sigma)} = \frac{1}{1-t+(1-x)\sum_{n=2}^{\infty}\frac{t^n}{n!}\sum_{w\in J_\tau,||w||=n}(-1)^{\overline{\ell}(w)}|\mathcal{P}^{\tau}_{w}|}.$$

The same method of proof can give several extension of Kitaev's theorem.

Let $\Upsilon\subseteq S_j$. We say that a permutation $\sigma=\sigma_1\cdots\sigma_n$ has an Υ-match starting at position $i+1$ if $\text{red}(\sigma_{i+1}\cdots\sigma_{i+j})\in\Upsilon$. For example, if $\sigma = 5\ 3\ 4\ 1\ 6\ 2$ and $\Upsilon=\{3\ 1\ 2, 2\ 1\ 3\}$, then σ has an Υ-match starting at positions 1 since $\text{red}(5\ 3\ 4)=3\ 1\ 2$ and an Υ-match starting at position 3 since $\text{red}(4\ 1\ 6)=2\ 1\ 3$. For any given permutation σ, we define Υ-*nlap*(σ) to be the maximum number of nonoverlapping Υ-matches that occur in σ. Let

$$\text{Mch}_\Upsilon(\sigma)=\{i:\text{red}(\sigma_{i+1}\cdots\sigma_{i+j})=\tau\}\qquad\text{and}$$
$$I_\Upsilon=\{1\le i<j:\text{there exist }\sigma\in S_{j+i}\text{ such that }\text{Mch}_\tau(\sigma)=\{0,i\}\}.$$

For example, it is easy to see that if $\Upsilon=\{3\ 1\ 2,2\ 1\ 3\}$, then $I_\Upsilon=\{2\}$.

Let I^*_Υ be the set of all words with letters in the set I_Υ. We let ϵ denote the empty word. If $w = w_1\cdots w_n\in I^*_\tau$ is word with n-letters, we define

$$\ell(w)=n,\qquad \sum w=\sum_{i=1}^{n}w_i,\qquad\text{and}\qquad ||w||=j+\sum w.$$

In the special case where $w=\epsilon$, we let $\ell(w)=0$ and $\sum w=0$. Let

$$A_\Upsilon=\{w\in I^*_\Upsilon:\ell(w)\ge 2\text{ and }\sum w<j\}\qquad\text{and}$$
$$B_{u,\Upsilon}=\{w_1\cdots w_n\in I^*_\Upsilon:\sum w_2\cdots w_n+\sum u<j\le\sum w_1\cdots w_n+\sum u\}$$

for each word $u\in I^*_\Upsilon$ with $\sum u<j$.

We also set

$$\mathcal{P}^{\Upsilon}_{w} = \{\sigma \in S_{||w||} : \mathrm{Mch}_{\Upsilon}(\sigma) = \{0, w_1, w_1+w_2, \ldots, w_1+w_2+\cdots+w_n\}\}.$$

For example, if $\Upsilon = \{3\ 1\ 2, 2\ 1\ 3\}$, then permutation $\sigma = 2\ 1\ 4\ 3\ 7\ 6\ 9\ 8\ 5\ \ 11\ \ 10$ has $\mathrm{Mch}_{\Upsilon}(\sigma) = \{0, 2, 4, 7\}$ so that σ is an element of $\mathcal{P}^{\Upsilon}_{2\ 2\ 3}$.

Form a new alphabet

$$K_{\Upsilon} = \{\overline{u} : u \in I_{\Upsilon}\} \cup \{\overline{w} : w \in A_{\Upsilon}\}.$$

We define the natural map $\Psi : K^*_{\Upsilon} \to I^*_{\Upsilon}$ such that $\Psi(\epsilon) = \epsilon$ and $\Psi(\overline{w_1} \cdots \overline{w_n}) = w_1 \ldots w_n$. Define $\overline{J}_{\Upsilon}$ recursively as follows.

(1) $\epsilon \in \overline{J}_{\Upsilon}$.
(2) $\overline{v} \in \overline{J}_{\Upsilon}$ for all $v \in I_{\Upsilon}$.
(3) If $\overline{w_1} \cdots \overline{w_n} \in \overline{J}_{\Upsilon}$, then $\overline{u}\ \overline{w_1} \cdots \overline{w_n} \in \overline{J}_{\Upsilon}$ for all $u \in B_{w_1,\Upsilon}$.
(4) The only words in $\overline{J}_{\Upsilon}$ are the result of applying one of the above rules.

Let $J_{\Upsilon} = \Psi(\overline{J}_{\Upsilon})$. Then once again, for each $w \in J_{\Upsilon}$, we can construct the unique word $\overline{u} = \overline{u_1} \cdots \overline{u}_t$ such that $\Psi(\overline{u}) = w$. We then let $\overline{\ell}(w) = \ell(\overline{u})$.

The number of inversions of a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ is $\mathrm{inv}(\sigma) = \sum_{i<\ell} \chi(\sigma_i > \sigma_\ell)$ where for any statement A, $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false. Given a letter σ_i in a permutation $\sigma_1 \cdots \sigma_n$, the number of inversions caused by σ_i is the number of indices $\ell > i$ such that $\sigma_i > \sigma_\ell$. This definition of inversions makes sense for any word of integers.

To understand the bivariate distribution of Υ-nlap with inv, it will be convenient to use standard notation from hypergeometric function theory. For $n \geq 1$ and $\lambda \vdash n$, let

$$[n]_q = \frac{1-q^n}{1-q} = q^0 + \cdots + q^{n-1}, \qquad [n]_q! = [n]_q \cdots [1]_q, \qquad \text{and} \qquad \begin{bmatrix} n \\ \lambda \end{bmatrix}_q = \frac{[n]_q!}{[\lambda_1]_q! \cdots [\lambda_\ell]_q!}$$

be the q-analogues of n, $n!$, and $\binom{n}{\lambda}$, respectively. By convention, we let $[0]_q = 0$ and $[0]_q! = 1$.

Let Υ be a subset of S_j for some $j > 2$. Let $f_q(n) = (-1)^n$ if $n = 0, 1$ and take

$$f_q(n) = (1-x) \sum_{\|\omega\|=n, \omega \in J_{\Upsilon}} (-1)^{\overline{\ell}(\omega)} \sum_{\sigma \in \mathcal{P}^{\Upsilon}_{\omega}} q^{inv(\sigma)}$$

otherwise. Define ξ_q as the ring homomorphism on Λ with the property that

$$\xi_q(e_n) = \frac{(-1)^n}{[n]_q!} f_q(n).$$

Essentially the same method of proof that was used to prove Kitaev's theorem can be used to prove the following.

Theorem 2.3. *For any set of permutations $\Upsilon \subseteq S_j$ where $j > 1$,*

$$A^{\Upsilon}_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n : Mch_{\Upsilon}(\sigma) = \emptyset} q^{inv(\sigma)} = \frac{1}{1 - t + \sum_{n=2}^{\infty} \frac{t^n}{[n]_q!} \sum_{w \in J_{\Upsilon}, ||w||=n} (-1)^{\overline{\ell}(w)} \sum_{\sigma \in \mathcal{P}^{\Upsilon}_{w}} q^{inv(\sigma)}}$$

and

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]!} \sum_{\sigma \in S_n} x^{\Upsilon-nlap(\sigma)} q^{inv(\sigma)} = \frac{1}{1 - t + (1-x) \sum_{n=2}^{\infty} \frac{t^n}{[n]_q!} \sum_{w \in J_{\Upsilon}, ||w||=n} (-1)^{\overline{\ell}(w)} \sum_{\sigma \in \mathcal{P}^{\Upsilon}_{w|}} q^{inv(\sigma)}}$$
$$= \frac{A^{\Upsilon}_q(t)}{(1-x) + x(1-t) A^{\Upsilon}_q(t)}.$$

To conclude this section we turn our attention to providing a result about pattern matches in multiple permutations. For $\Upsilon \subseteq S_j$, we define an a collection of permutations $\sigma^1, \ldots, \sigma^m \in S_n$ to have a Υ-common match at place i provided that there is a $\sigma \in \Upsilon$ such that $\mathrm{red}(\sigma^\ell_i \cdots \sigma^\ell_{i+j-1}) = \sigma$ for every $1 \leq \ell \leq m$. Let Υ-*commch*$(\sigma^1, \ldots, \sigma^m)$ be the total number of Υ-common matches and let Υ-*comnlap*$(\sigma^1, \ldots, \sigma^m)$ be the maximum number of nonoverlapping Υ-common matches in the collection of elements $\sigma^1, \ldots, \sigma^m \in S_n$. If Υ consists of a single permutation τ, then we use the notation τ-common match, τ-*commch*$(\sigma^1, \ldots, \sigma^m)$, and

τ-*comnlap*$(\sigma^1,\ldots,\sigma^m)$ for Υ-common match, Υ-*commch*$(\sigma^1,\ldots,\sigma^m)$, and Υ-*comnlap*$(\sigma^1,\ldots,\sigma^m)$ respectively.

When $\tau = 2\ 1$, the number of τ-common matches has been referred to as the number of common descents and was first studied by Leonard Carlitz, Richard Scoville and Theresa Vaughan [**CSV74, CSV76a, CSV76b**]. Later, this statistic was studied by Jean-Marc Fédou with Don Rawlings and Thomas Langley with Jeff Remmel; the latter authors used the same approach of manipulating the relationships between symmetric functions we are taking [**FR94, FR95, LR**]. This paper, however, marks the first time τ-common matches have been studied for τ other than the permutation 2 1.

Then we can define an obvious analogue of $\mathcal{P}_w^{\Upsilon}$ for m-tuples of permutations, $\mathcal{P}_w^{\Upsilon,m}$, to prove the following.

THEOREM 2.4. *For any set of permutations $\Upsilon \subseteq S_j$ where $j > 1$,*

$$A^{\Upsilon,\vec{q},m}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\prod_{i=0}^m [n]_{q_i}!} \sum_{(\sigma^1,\ldots,\sigma^m)\in S_n^m : \Upsilon\text{-}commch(\sigma^1,\ldots,\sigma^m)=0} \prod_{i=1}^n q_i^{inv(\sigma^i)}$$
$$= \frac{1}{1 - t + \sum_{n=2}^{\infty} \frac{t^n}{\prod_{i=1}^m [n]_{q_i}!} \sum_{w\in J_\Upsilon, ||w||=n} (-1)^{\overline{\ell}(w)} \sum_{(\sigma^1,\ldots,\sigma^m)\in \mathcal{P}_w^{\Upsilon,m}} \prod_{i=1}^m q_i^{inv(\sigma^i)}},$$

and

$$\sum_{n=0}^{\infty} \frac{t^n}{\prod_{i=1}^m [n]_{q_i}!} \sum_{(\sigma_1,\ldots,\sigma^m)\in S_n^m} x^{\Upsilon\text{-}comnlap(\sigma^1,\ldots,\sigma^m)} \prod_{i=1}^m q_i^{inv(\sigma^i)}$$
$$= \frac{1}{1 - t + (1-x)\sum_{n=2}^{\infty} \frac{t^n}{\prod_{i=1}^m [n]_{q_i}!} \sum_{w\in J_\Upsilon, ||w||=n} (-1)^{\overline{\ell}(w)} \sum_{(\sigma^1,\ldots,\sigma^m)\in \mathcal{P}_w^{\Upsilon,m}} \prod_{i=1}^m q_i^{inv(\sigma^i)}}$$
$$= \frac{A^{\Upsilon,\vec{q},m}(t)}{(1-x) + x(1-t)A^{\Upsilon,\vec{q},m}(t)}.$$

We should also note that our methods can be extended to give similar theorems for pattern matchings in words.

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