Enumeration of Reading's twisted Baxter permutations

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In this note we provide a bijection between the Baxter permutations Nathan Reading's twisted Baxter permutations.

The Baxter permutations were studied by Chung et. al. in [2], which famously introduced the idea of a generating tree as a way of studying hereditary classes of permutations. In terms of forbidden subsequences, the Baxter permutations are those which avoid $41\bar{3}52$ and $25\bar{3}14$, as first observed by Olivier Guibert [3].

In [4], Nathan Reading introuced a new class of permutations, which he calls *twisted Baxter permutations*, and which are characterized by avoidance of $45\bar{3}12$ and $25\bar{3}14$. In that paper, Reading notes that the Baxter permutations and the twisted Baxter permutations are equinumerous through length $n = 15$.

We will now prove that these two classes of permutations are indeed equinumerous.

1 Baxter permutations

In [1], Mireille Bousquet-Melou describes the construction of the Baxter permutations as follows:

> It is not very hard to see that $(n + 1)$ has to be inserted:
>
> - either just before a left-to-right maximum of σ, or
> - just after a right-to-left maximum of σ.
>
> We are thus led to introduce two additional statistics, namely the number of left-to-right maxima and the number of right-to-left maxima of σ.

We will supply the details in this proof, en route to describing a similar characterisation of the new class of permutations.

Let the left-to-right maxima have values $l_1, l_2, \dots l_p = n$, and the right-to-left maxima have values $n = r_q, r_{q-1}, \dots r_1$.

Because one of the Baxter conditions is the reverse of the other, the situation is left/right symmetric and we need only consider insertions on one side of n. We will choose the righthand side.

Suppose $(n+1)$ is inserted between two right-to-left maxima, r_j and r_{j-1}, not immediately after r_j, but instead immediately after some intervening entry s. Because s is not in the sequence of right-to-left maxima, it must be smaller than r_{j-1}, so the sequence $r_j, s, (n+1), r_{j-1}$ is of the forbidden type 3142. Moreover, it is of type $41\bar{3}52$ as s and $(n+1)$ are adjacent.

On the other hand, suppose there is no intervening s, so that we have $r_j, (n+1), r_{j-1}$, with the first two of these adjacent. We must check that this insertion cannot create either a $41\bar{3}52$ or a $25\bar{3}14$ where none existed before.

If a $\langle 41\bar{3}52 \rangle = a, b, (n+1), d$ has been created, then $a > r_j$, or else $a, b, r_j, d = \langle 41\bar{3}52 \rangle$ as well. But also $r_j > r_{j-1} \geq d$, because r_{j-1} is the largest element to the right of $(n+1)$. This means that $a, b, r_j, (n+1), d = \langle 41352 \rangle$, a contradiction.

If a $\langle 25\bar{3}14 \rangle = d, (n+1), b, a$ has been created, then $a \leq r_{j-1} < r_j$, so d, r_j, b, a was already of type $25\bar{3}14$.

This completes the consideration of insertions to the right of n. As already remarked, the situation to the left of n is identical.

2 Twisted Baxter permutations

We now look at what changes need to be made to adapt this to Reading's new class of permutations, $S_n(45\bar{3}12, 25\bar{3}14)$.

We apply the same labels to the left-to-right maxima and right-to-left maxima. We claim that the active insertion sites are now:

- either just before a left-to-right maximum of σ, or
- just before a right-to-left maximum of σ, or
- after the final element of σ.

This last case can be thought of as occurring between the final element, r_1, and a dummy zero element, r_0. In this sense, the single insertion point between r_j and r_{j-1} ($j = 1 \ldots q$) is always as far right as possible, just as in the Baxter permutations it was always as far left as possible.

We begin on the righthand side.

Suppose $(n+1)$ is inserted just to the left of a right-to-left non-maximum, s, located between r_j and r_{j-1}. Then $r_j, (n+1), s, r_{j-1}$ is a $45\bar{3}12$.

On the other hand, suppose $(n+1)$ is inserted just to the left of r_{j-1}.

If this creates a $\langle 45\bar{3}12\rangle = a, (n+1), c, d$, then $a < r_{j-1}$ implies a, r_{j-1}, c, d is a pre-existing $45\bar{3}12$, while $a > r_{j-1}$ implies $a, (n+1), r_{j-1}, c, d$ is of type 45312.

If it creates a $\langle 25\bar{3}14\rangle = d, (n+1), b, a$, note that a is not r_{j-1}, because $(n+1)$ and r_{j-1} are adjacent. Thus $a \leq r_{j-2}$, the largest element to the right of r_{j-1}, and therefore d, r_{j-1}, b, a was already of type $25\bar{3}14$.

Now consider insertions on the lefthand side.

Suppose $(n+1)$ is inserted to the left of a left-to-right non-maximum, t, located between l_{j-1} and l_j. Then $l_{j-1}, (n+1), t, l_j$ is a $25\bar{3}14$.

Now suppose $(n+1)$ is inserted just to the left of l_j.

If this creates a $25\bar{3}14 = d, (n+1), b, a$, then $a < l_j$ implies d, l_j, b, a is a pre-existing $25\bar{3}14$, while $a > l_j$ implies $d, (n+1), l_j, 1, 4$ is of type 25314.

If it creates a $\langle 45\bar{3}12\rangle = a, (n+1), c, d$, then $a \leq l_{j-1} < l_j$, so a, l_j, c, d is a pre-existing $45\bar{3}12$.

3 Conclusion

To see that this establishes a bijection between the two classes of permutation, note that in each case there is exactly one insertion point between l_i and l_{i+1}, and that insertion into it produces a permutation with $i+1$ left-to-right maxima and $q+1$ right-to-left maxima. Similarly there is exactly one insertion point between r_{j+1} and r_i, insertion into which produces a permutation with $p+1$ left-to-right maxima and $j+1$ right-to-left maxima. This induces an isomorphism between the two-parameter generating trees.

4 Bibliography

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