Determining lower bounds for packing densities of non-layered patterns using weighted templates

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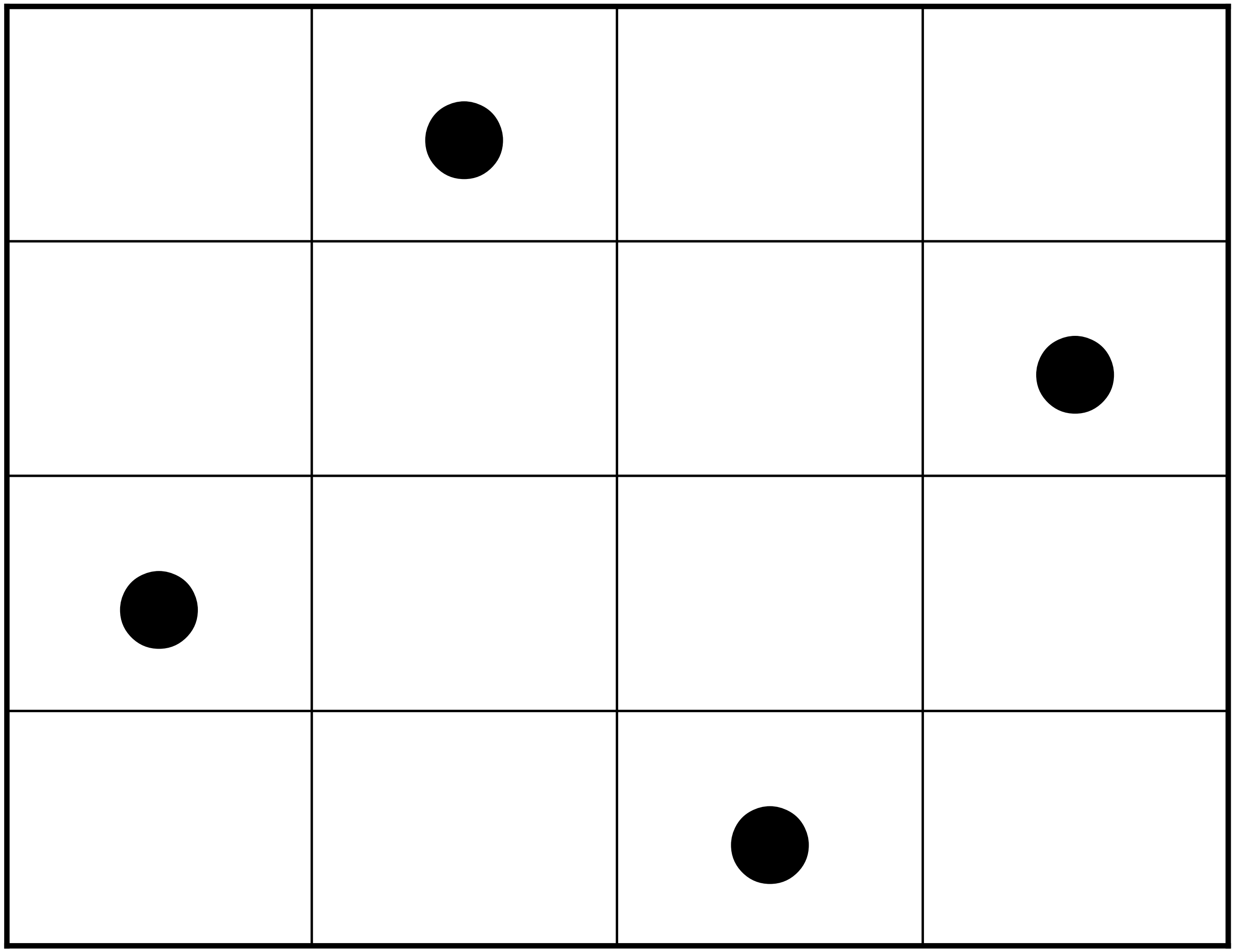
I) Background

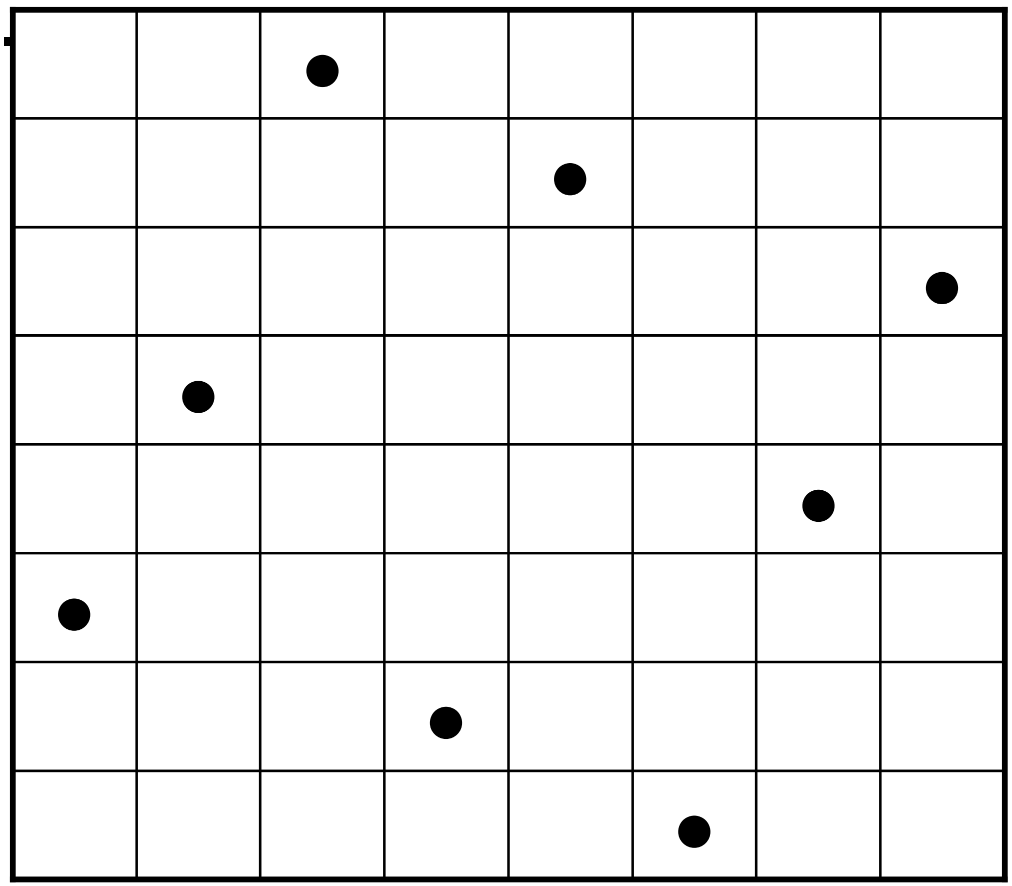
The topic of packing densities of *layered* patterns is fairly well covered. However the topic of packing densities of *non-layered* patterns is not as well understood.

From Albert, Atkinson, Handley, Holton, and Stromquist [AAHHS]: "On packing densities of permutations," EJC 9(2002)

Consider the non-layered pattern 2413:



A lower bound for the packing density of 2413 can be determined in the following way: The permutation 35827146 contains a relatively high number (17) of occurrences of 2413.



Consider the permutations of the form $\sigma = \sigma_3\sigma_5\sigma_8\sigma_2\sigma_7\sigma_1\sigma_4\sigma_6$, where for each i , the blocks $\sigma_i < \sigma_{i+1}$ and $|\sigma_i| = |\sigma_{i+1}|$ and where each σ_i is recursively structured in the same way. It is easy to see that $| \sigma |$ equals a multiple of 8, call it n.

There are several ways for a given pattern to occur in a permutation. We will use only two of them in finding the probability p_n that 2413 will occur in the aforementioned σ .

1. All four points of the 2413 occurrence lie in a single σ_i: $8(1/8)^4\, p_{n/8}$

2. Each of the four points of the 2413 occurrence lies in a distinct σ_i: $17 \times 4! \times (1/8)^4$

Thus the probability p_n that 2413 will occur in σ will be: $p_n = 8\ (1/8)^4\ p_{n/8} + 17 \times 4! \times (1/8)^4$

As n approaches infinity, p_n approaches a limit p, yielding the equation

$$p = 8\ (1/8)^4\ p + 17 \times 4! \times (1/8)^4$$

and its solution, unweighted

$p = 51/511 \approx 0.099804305283837573\mathbf{4}$

which is a lower bound for the actual packing density of 2413.

II) Weighted Template based on multiplicity

In the 17 occurrences of 2413 in 35827146, the points 3, 8, 1, & 6 appeared 9 times and 5, 2, 7, & 4 appeared 8 times.

		x=9					
				y=8			
							x=9
	y=8						
						y=8	
x=9							
			y=8				
					x=9		

We apply this to the recursively constructed σ mentioned in I). σ_3, σ_8, σ_1, and σ_6 each will have a weight of 9/68 and σ_5, σ_2 , σ_7 , and σ_4 each will have a weight of 8/68.

The weighted probability equation becomes

$$p = [4(9/68)^4 + 4(8/68)^4]\, p + 4![9^4 + 4(9^3\times 8) + 8(9^2\times 8^2) + 4(9\times 8^3)]/68^4 .$$

Thus <u>weighted</u> $p \approx 0.1009914920969121 25$

& <u>unweighted</u> $p \approx 0.0998043052838375734$.

Now let's consider the permutation 468(12)3(11)2(10)1579 and the number of 2413 patterns in it. There are 86 of them.

			31								
					29						
							26				
											31
		26									
										29	
	29										
									26		
31											
				26							
						29					
								31			

Unweighted:

Our probability equation becomes

$p = 12\,(1/12)^4\, p + 86 \times 4! \times (1/12)^4$.

Weighted:

Our recursively constructed σ has the form $\sigma_4\sigma_6\sigma_8\sigma_{12}\sigma_3\sigma_{11}\sigma_2\sigma_{10}\sigma_1\sigma_5\sigma_7\sigma_9$ and the weights are 31/344 for σ_4, σ_{12}, σ_1, & σ_9, 29/344 for σ_6, σ_{11}, σ_2, & σ_7 and 26/344 for σ_8, σ_3, σ_{10}, & σ_5.

Using the permutation 468(12)3(11)2(10)1579,

we find the following probabilities:

<u>weighted</u> $p \approx 0.10137416835977531252$
<u>unweighted</u> $p \approx 0.099594672843080 49$.

Thus far, the weighted probability is yielding a better lower bound for the packing density of 2413.

III) Optimally Weighted Template

Now we will write the general probability equation as a multivariable function with variables replacing of the multiplicity weights

and will find the optimal weights by locally maximizing the function.

Example for 2413 in 35827146:

The general probability equation is
$f(x,y)=24[x^4+4x^3y+8x^2y^2+4xy^3] / [1-4x^4-4y^4]$.

Using Mathematica to maximize f with the constraints $4x + 4y = 1$ and x & $y > 0$, we find that the optimal weights are
$x \approx 15544748590173554$ and
$y \approx 0.09455251409826446$.

These optimal weights provide us with an improved lower bound for the packing density of 2413 as compared below:

Optimally weighted:

0.10247328135488704

Weighted:

0.10099149209691215

Unweighted:

0.09980430528383757734

Example for 2413 in 468(12)3(11)2(10)1579:

The general probability equation is

$$f(x,y,z) = 24[x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 + 4x^3z + 16x^2yz + 16xy^2z + 4y^3z + 8x^2z^2 + 12xyz^2 + 6y^2z^2 + 4xz^3] / [1 - 4(x^4 + y^4 + z^4)]$$

Using Mathematica to maximize f with the constraints $4x+4y+4z=1$ and x, y, & $z > 0$, we find that the optimal weights are

$x \approx 0.12461011912365183$,

$y \approx 0.06830111111514646$, and

$z \approx 0.05708876976120 17$

These new optimal weights provide us with an improved lower bound for the packing density of 2413 as compared below:

Optimally weighted:

0.10381609308698811

Weighted:

0.1013741683597753125218536

Unweighted:

0.09959467284308049

IV) Weighted Template works for other patterns.

Example:

There are 36 occurrences of 51324 in 9(10)15326478.

Multiplicities are shown to the right.

	18								
18									
									14
								14	
						18			
			16						
							18		
				18					
					18				
		28							

Unweighted:

The probability equation is

$$p = 10\,(1/10)^5\, p + 36 \times 5! \times (1/10)^5$$

and yields $p \approx 0.043204320432043204$ as a lower bound for the packing density of 51324.

Weighted:

p = $[6a^5+2b^5+c^5+d^5]p + 120[8a^4b+4a^4d+8a^3bd+4a^3cd+12a^2bcd]$, where a = 18/180, b = 14/180, c = 16/180, and d = 28/180.

From this we get a weighted lower bound of $p \approx 0.0486795067329173231137$ compared to the unweighted lower bound of $p \approx 0.043204320432043204$.

The general probability function is

$f(a,b,c,d) = 120[8a^4b+4a^4d+8a^3bd+4a^3cd+12a^2bcd] / [1-6a^5-2b^5-c^5-d^5]$.

Again Mathematica is used to maximize the function with the constraints $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$, and $6a+2b+c+d=1$. In trying to find the optimal weights using Mathematica, we find that the program wants to assign the variable a to be significantly close to 0, $b < 0$, $c < 0$, and $d = 1$.

Attempts to correct this are as follows:

1) Change to $a \geq 0.01$. This results in optimal weights of $a \approx 0.150019$, $b = 0$,$c = 0$,
$d \approx 0.099885$ and a lower bound of 0.14589.

2) Change to $b \geq 0.01$. This results in optimal weights of $a \approx 0.147018$, $b = 0.01$,
$c \approx 1.96282\times10^{-10}$, $d \approx 0.0978919$ and a lower bound of 0.131865.

3) Change to $c \geq 0.01$. This results in optimal weights of $a \approx 0.146642$, $b = 1.49143\times10^{-8}$, $c = 0.0100075$, $d \approx 0.110139$
and a lower bound of 0.145192.

From this data, we can infer that the permutation 9(10)15326478 was not the best template for the pattern 51324 and a better one may found by eliminating either the “b” blocks alone or both “b” & “c” blocks.