SOME ORDER-THEORETIC PROPERTIES OF THE MOTZKIN AND SCHRÖDER FAMILIES

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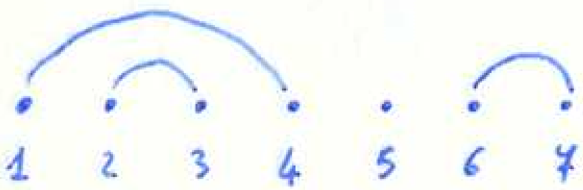
THE CATALAN FAMILY [BBFP]

$\mathcal{D}_n$: DYCK PATHS OF LENGTH $2n$

$NC(n)$: NONCROSSING PARTITIONS OF AN n-SET



32/41/5/76

$S_n(312)$: 312-AVOIDING PERMUTATIONS OF AN n-SET

3241576

[BBFP]: E. BARCUCCI, A. BERNINI, L. F., M. PONETI, A DISTRIBUTIVE LATTICE STRUCTURE CONNECTING DYCK PATHS, NONCROSSING PARTITIONS AND 312-AVOIDING PERMUTATIONS, ORDER 22 (2005) 311-328

THE CATALAN FAMILY [BBFP]

$[\mathcal{D}_n ; \leq]$

DYCK LATTICE OF ORDER n

•) DISTRIBUTIVE

•) RANK : $r(P) = \frac{A(P) - n}{2}$

$[NC(n) ; \leq]$

BRUHAT NONCROSSING PARTITION LATTICE

$21|43|5|6|7 \leq 32|41|5|76$

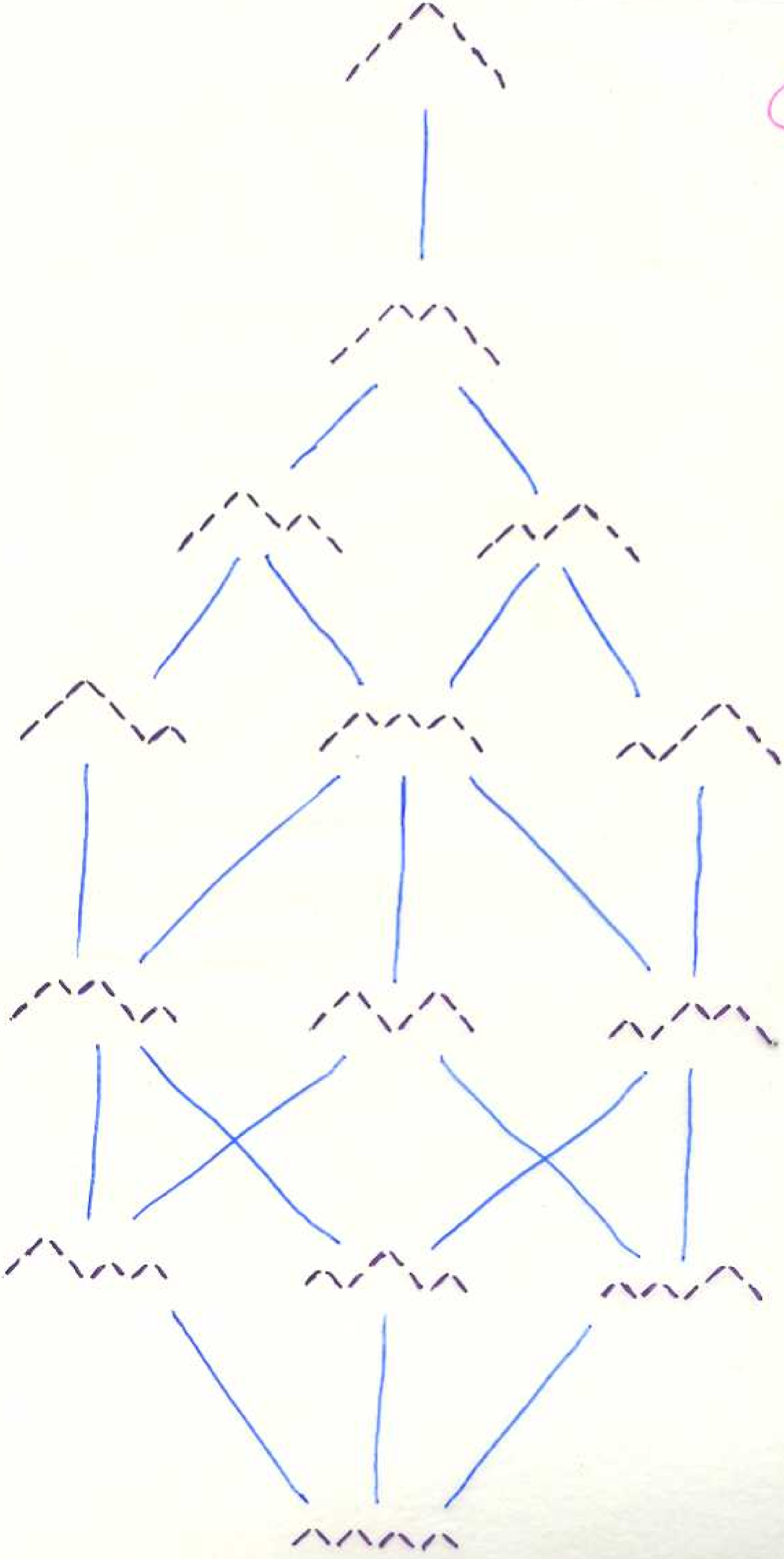
$\tilde{\pi} \leq \rho$ WHEN $\max(\tilde{\pi}) \leq \max(\rho)$

$[S_n(312) ; \leq]$

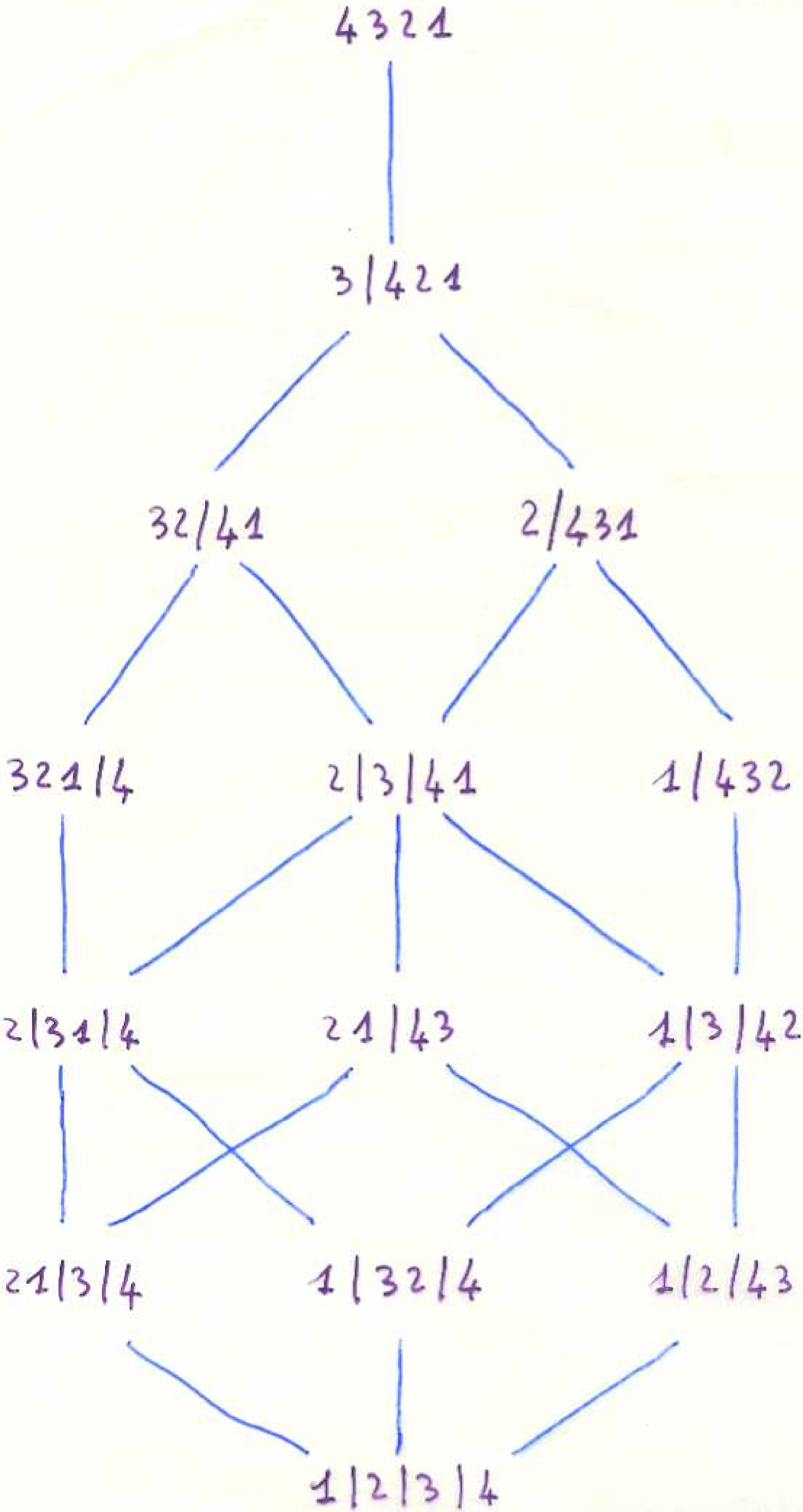
POSET INDUCED BY THE STRONG BRUHAT ORDER ON S_n

IT IS A DISTRIBUTIVE LATTICE !

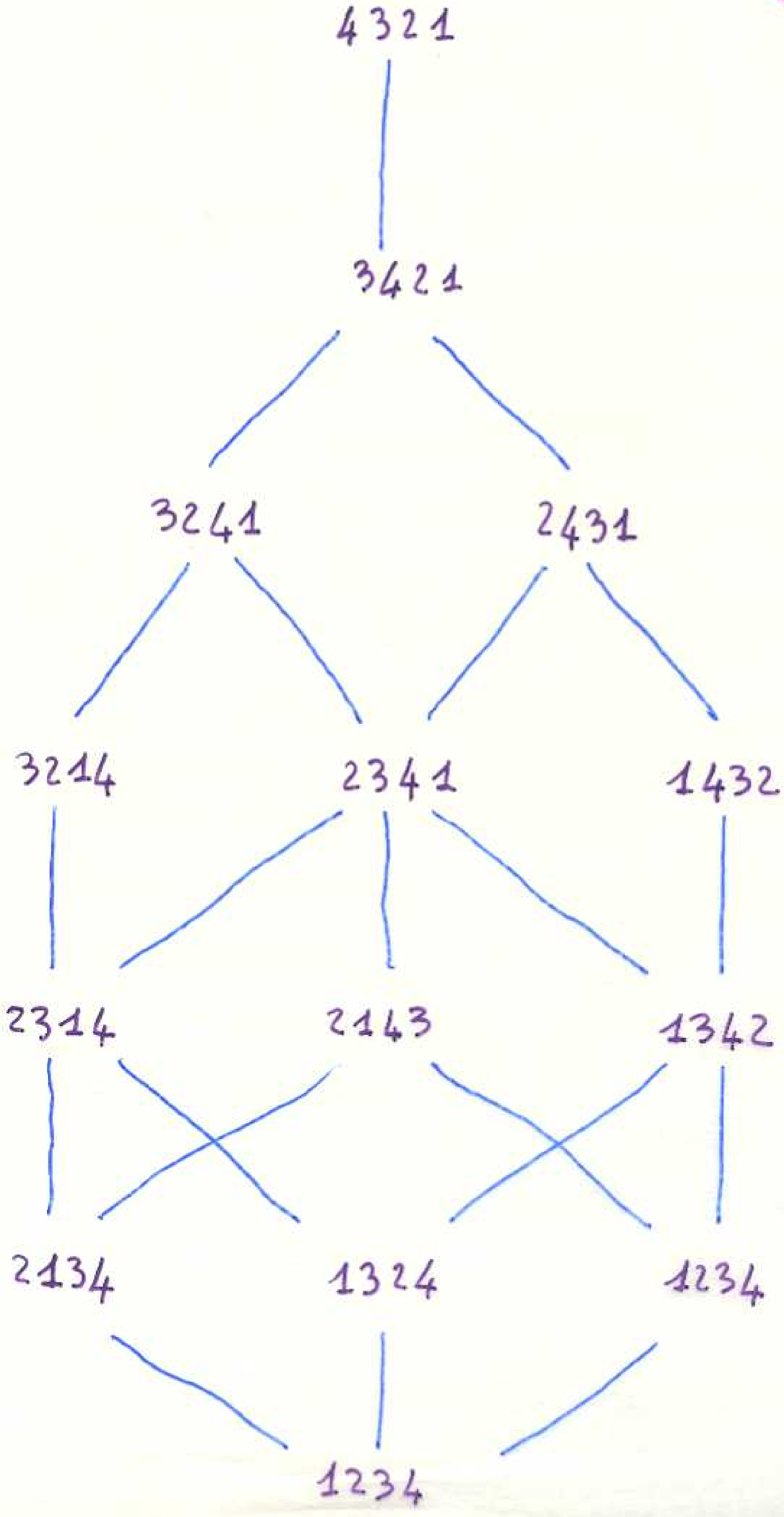
ω_4



NC (4)


4321
3|421
32|41
2|431
321|4
2|3|41
1|432
2|31|4
21|43
1|3|42
21|3|4
1|32|4
1|2|43
1|2|3|4

$S_4(312)$


4321
3421
3241
2431
3214
2341
1432
2314
2143
1342
2134
1324
1234
1234

DOES SOMETHING SIMILAR

HOLD

FOR THE

MOTZKIN AND SCHRÖDER

FAMILIES

?

THE MOTZKIN FAMILY

$[\mathcal{M}_n ; \leq]$: MOTZKIN LATTICE OF ORDER n

IT IS A DISTRIBUTIVE LATTICE

A BIJECTION (S. ELIZALDE, T. MANSOUR, RESTRICTED MOTZKIN PERMUTATIONS, MOTZKIN PATHS, CONTINUED FRACTIONS, AND CHEBYSHEV POLYNOMIALS, DISCRETE MATH., 305 (2005) 170-189)

$\mathcal{D}_n^{(3)}$: DYCK PATHS OF LENGTH $2n$ WITHOUT THREE CONSECUTIVE DOWN STEPS

$$f : \mathcal{D}_n^{(3)} \longrightarrow \mathcal{M}_n$$

$$: \quad U \longmapsto U$$

$$UD \longmapsto H$$

$$UDD \longmapsto D$$

PROPOSITION: φ IS AN ORDER ISOMORPHISM

$MNC(n)$: NONCROSSING PARTITIONS OF AN n-SET WHOSE BLOCKS HAVE CARDINALITY AT MOST 2 (PARTIAL MATCHINGS...)

MOTZKIN NONCROSSING PARTITIONS

$[MCN(n);\leq]$ AS A SUBPOSET OF $[NC(n);\leq]$ TURNS OUT TO BE A DISTRIBUTIVE LATTICE ISOMORPHIC TO $[\mathcal{M}_n;\leq]$

THEOREM: (COVERING RELATION OF $MNC(n)$)

$$\tilde{\pi}, \rho \in MNC(n)$$

$$\tilde{\pi} \prec \rho$$

$$\Updownarrow$$

ρ IS OBTAINED FROM $\tilde{\pi}$ BY MOVING THE MINIMUM OF SOME BLOCK B OF $\tilde{\pi}$ INTO THE BLOCK $\tilde{B}$ CONTAINING $\beta = \max B + 1$ IF $\beta = \min \tilde{B}$. IN THIS CASE, EITHER:

1) KEEP β INSIDE $\tilde{B}$, IF $|\tilde{B}| = 1$, OR

2) ADD A NEW BLOCK $\hat{B} = \{\beta\}$, IF $|\tilde{B}| = 2$.

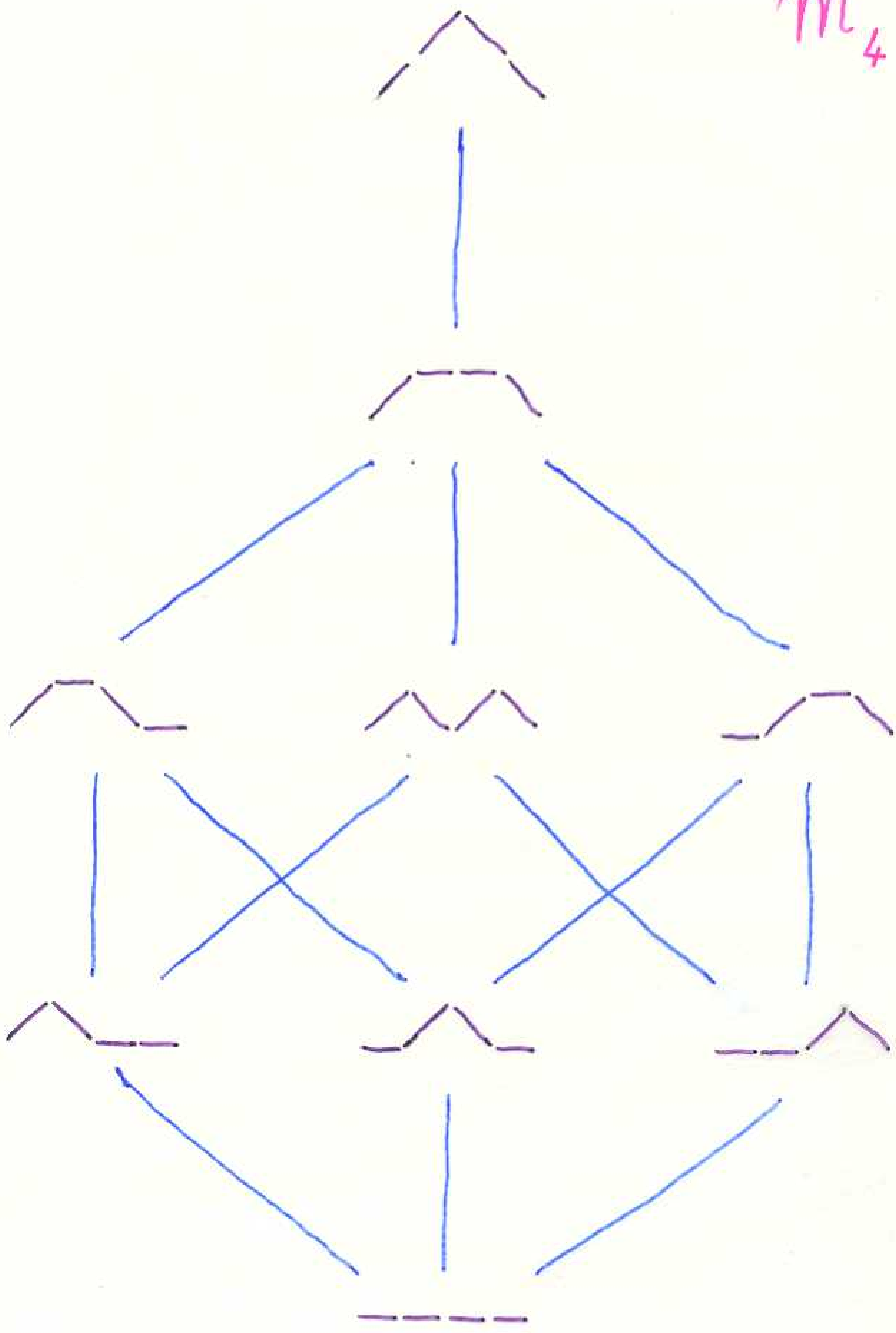
EXAMPLE

$$
\begin{tikzcd}[column sep=small]
2|3|4|65|71|8 \arrow[dr, no head] & & 2|31|65|7|84 \arrow[dl, no head] \\
& 2|31|65|74|8 &
\end{tikzcd}
$$

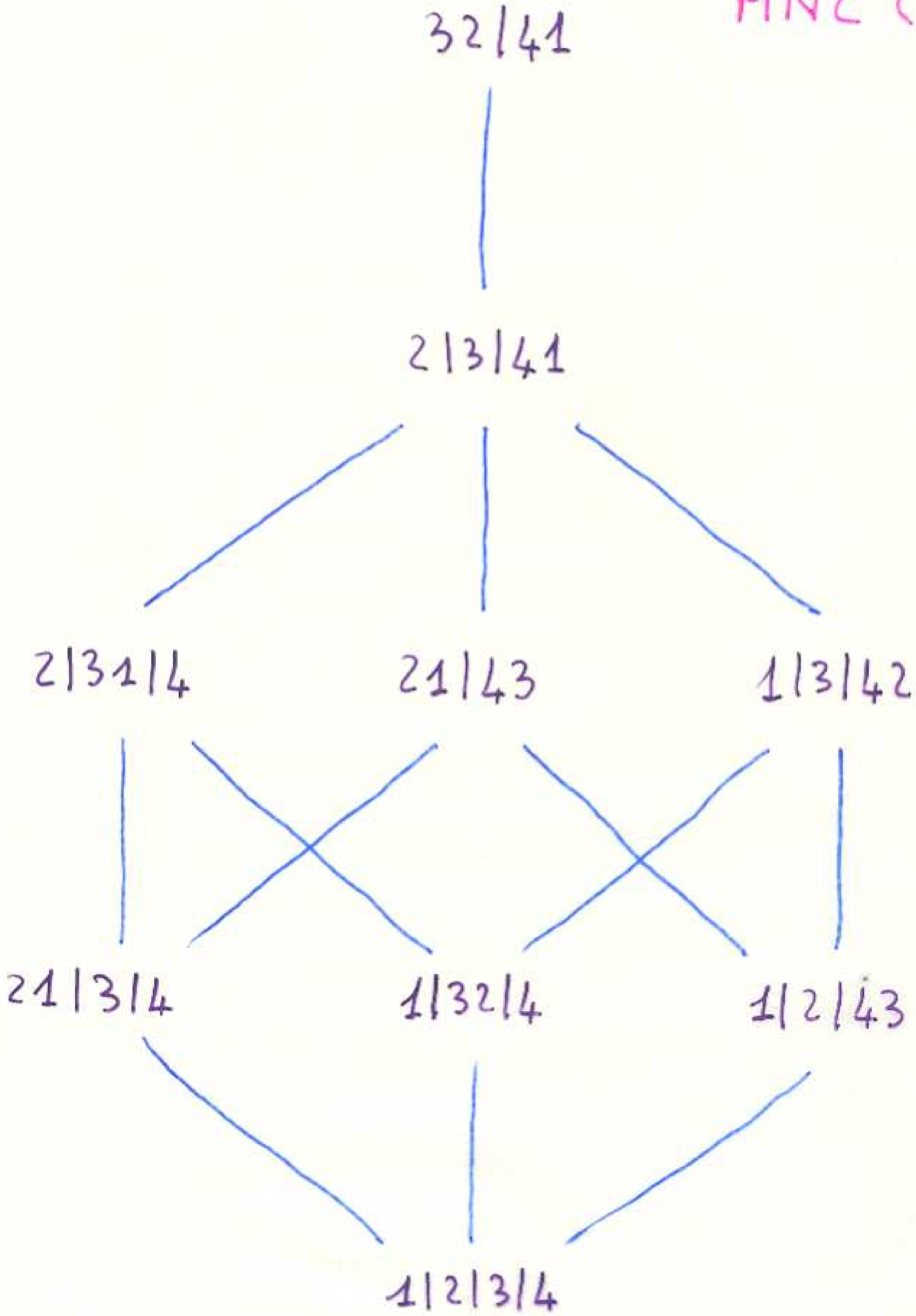
USING THE BAR-REMOVING BIJECTION:

THEOREM: $S_n(3\text{-}21, 31\text{-}2)$ IS A DISTRIBUTIVE SUBLATTICE OF THE POSET S_n (WITH THE STRONG BRUHAT ORDER) WHICH IS ISOMORPHIC TO $\mathcal{M}_n$.

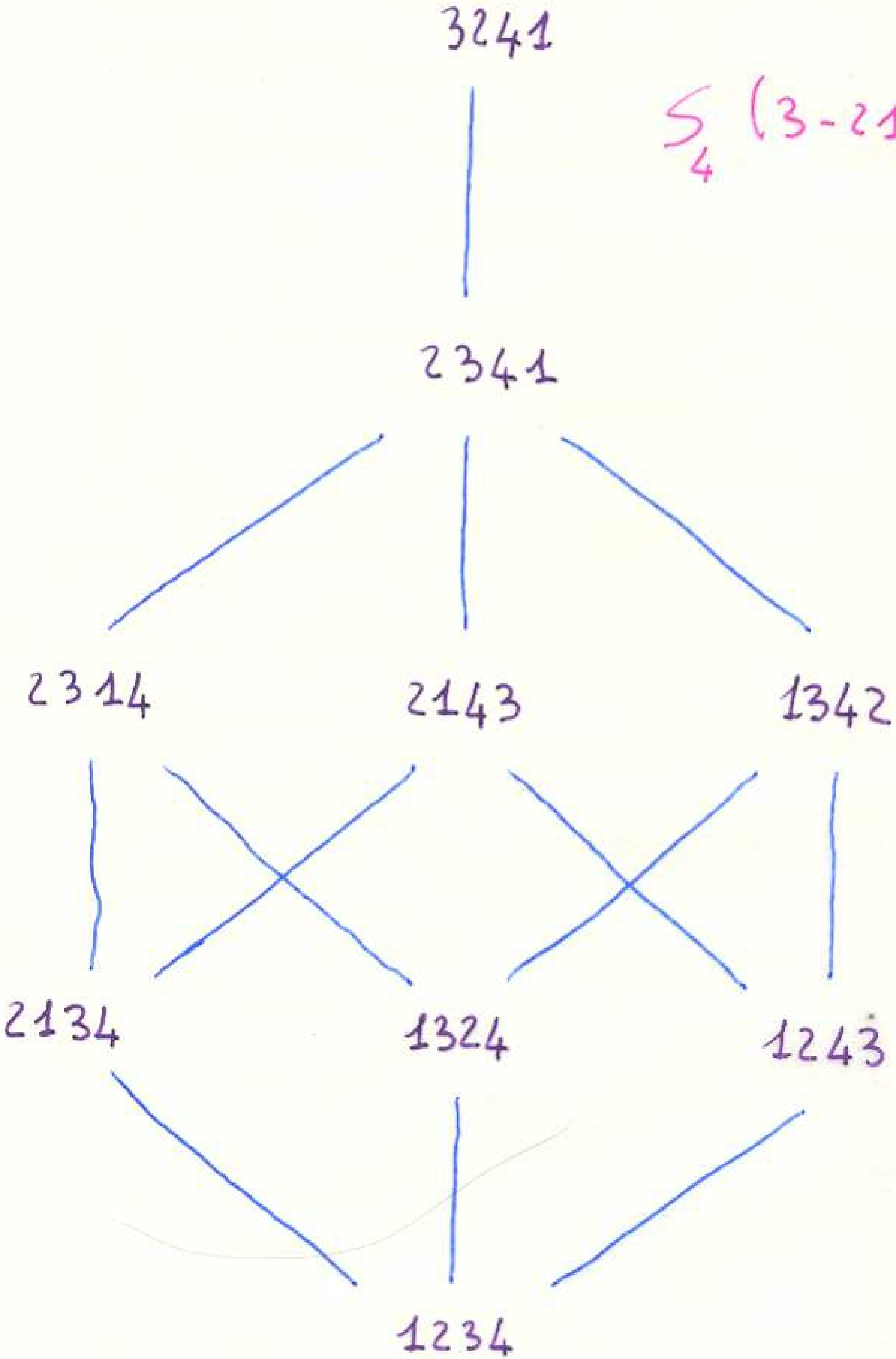
m_4



MNC (4)



$S_4(3\text{-}21, 31\text{-}2)$



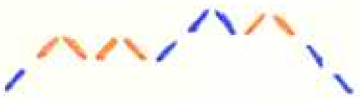
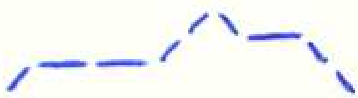
THE SCHRÖDER FAMILY

$\overline{\mathcal{D}}_n$: DYCK PATHS OF LENGTH $2n$ WITH BICOLOURED PEAKS

$\mathcal{S}_n$: SCHRÖDER PATHS OF LENGTH $2n$

AN OBVIOUS BIJECTION :

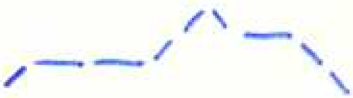
$$g : \overline{\mathcal{D}}_n \longrightarrow \mathcal{S}_n$$

 $\longmapsto$ 

$\overline{NC}(n)$: NONCROSSING PARTITIONS OF AN n-SET SUCH THAT THE MAXIMA OF THE BLOCKS ARE BICOLOURED

AN OBVIOUS BIJECTION :

$$g' : \mathcal{S}_n \longrightarrow \overline{NC}(n)$$

 $\longmapsto$ $\overline{2} \mid \overline{3} \mid 5 \mid \overline{6}\,4\,1$

$[\mathcal{S}_n ; \leq]$: SCHRÖDER LATTICE OF ORDER n

IT IS A DISTRIBUTIVE LATTICE

TRANSFER THE ABOVE ORDER STRUCTURE ON $\overline{NC}(n)$ ALONG g' :

$[\overline{NC}(n) ; \leq]$: SCHRÖDER NONCROSSING PARTITION LATTICE OF ORDER n.

THEOREM : (COVERING RELATION OF $\overline{NC}(n)$)

$$\tilde{\pi}, \rho \in \overline{NC}(n)$$

$$\tilde{\pi} \prec \rho$$

$$\Updownarrow$$

ρ IS OBTAINED FROM $\tilde{\pi}$ EITHER BY

1) REMOVING THE COLOUR FROM AN ELEMENT OF $\tilde{\pi}$, OR

2) MOVING THE MINIMUM OF SOME BLOCK B OF $\tilde{\pi}$ INTO THE BLOCK $\tilde{B}$ CONTAINING THE ELEMENT $\beta = \max B + 1$ ONLY WHEN β IS NOT COLOURED ; MOREOVER :

a) IF $\beta = \max \tilde{B}$, THEN KEEP β INSIDE $\tilde{B}$ AND COLOUR IT ;

b) IF $\beta \neq \max \tilde{B}$, THEN ADD THE BLOCK $\hat{B} = \{\bar{\beta}\}$

EXAMPLE:

$$
\begin{tikzcd}[column sep=small]
543|62|871|\bar{9} \arrow[drr, dash] & \bar{5}4|\bar{6}32|871|\bar{9} \arrow[dr, dash] & & \bar{5}43|6|\bar{7}|821|\bar{9} \arrow[dl, dash] & \bar{5}43|62|871|9 \arrow[dll, dash] \\
& & \bar{5}43|62|871|\bar{9} & &
\end{tikzcd}
$$

$\overline{S}_n$: COLOURED PERMUTATIONS OF AN n-SET

USING THE BAR-REMOVING BIJECTION:

THEOREM: $\overline{NC}(n) \longleftrightarrow \overline{S}_n(312, \bar{2}\bar{1}, 2\bar{1}, \bar{3}12)$

(SEE ALSO EGGE)

$\overline{S}_n(312, \bar{2}\bar{1}, 2\bar{1}, \bar{3}12)$ CAN BE EQUIPPED WITH A DISTRIBUTIVE LATTICE STRUCTURE

THEOREM: (COVERING RELATION)

$$\tilde{\pi}, \rho \in \bar{S}_n(312, \bar{2}\bar{1}, 2\bar{1}, \bar{3}12)$$

$$\tilde{\pi} \prec \rho$$

$$\Updownarrow$$

ρ IS OBTAINED FROM $\tilde{\pi}$ EITHER BY:

1) REMOVING THE COLOUR FROM AN ELEMENT OF $\tilde{\pi}$, OR

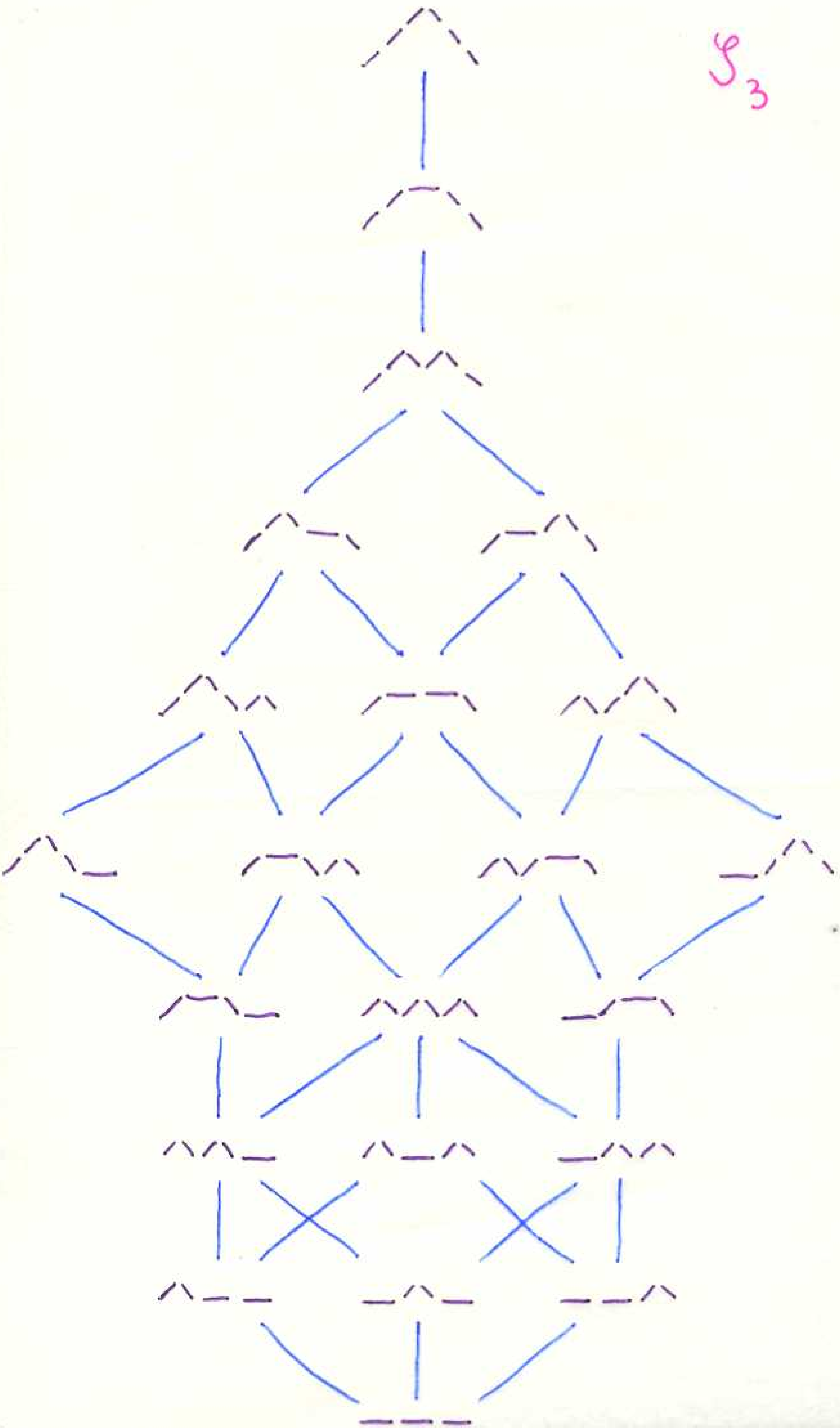
2) INTERCHANGING THE LAST ELEMENT α OF A DESCENT OF $\tilde{\pi}$ WITH β, WHERE $\beta-1$ IS THE FIRST ELEMENT OF THAT DESCENT, AND COLOURING β; THIS LAST OPERATION CAN BE PERFORMED EXCLUSIVELY WHEN α AND β ARE BOTH UNCOLOURED.

REMARK: $[\bar{S}_n(312, \bar{2}\bar{1}, 2\bar{1}, \bar{3}12); \leq]$ IS <u>NOT</u> A SUBPOSET OF $\bar{S}_n$ WITH THE STRONG BRUHAT ORDER.

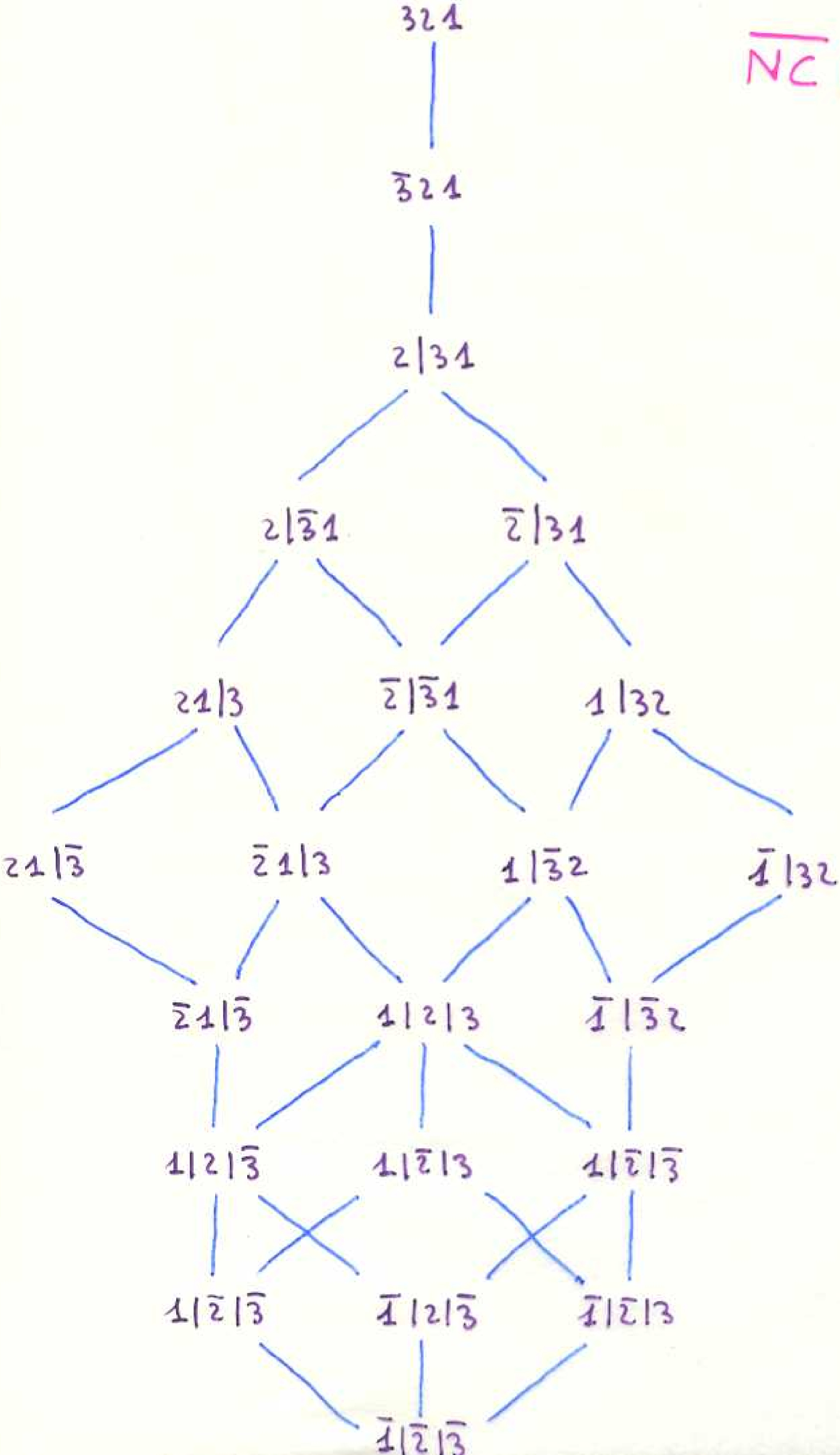
CONCERNING THE RANK OF $\bar{S}_n(312, \bar{2}\bar{1}, 2\bar{1}, \bar{3}12)$:

PROPOSITION: $r(\tilde{\pi}) = 2 \cdot |inv(\tilde{\pi})| + nc(\tilde{\pi})$

$\mathcal{S}_3$



$\overline{NC}(3)$



$\bar{S}_3(312, \bar{2}\bar{1}, 2\bar{1}, \bar{3}12)$