Forest-like permutations

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Outline

I. "Locally factorial" permutations [Woo-Yong 05]

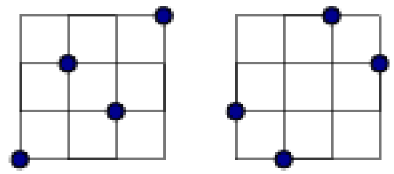
II. Characterization: "locally factorial" $\equiv$ forest-like

III. Enumeration

IV. Open questions

“Smooth” permutations

Theorem [Lakshmibai-Sandhya 90] : The Schubert variety X_π is smooth iff π avoids 1324 and 2143.



Generating function [Haiman 92, 06]:

$$S(x) = \sum_{n \geq 1} s_n x^n = x \frac{1 - 5x + 4x^2 + x\sqrt{1-4x}}{1 - 6x + 8x^2 - 4x^3}.$$

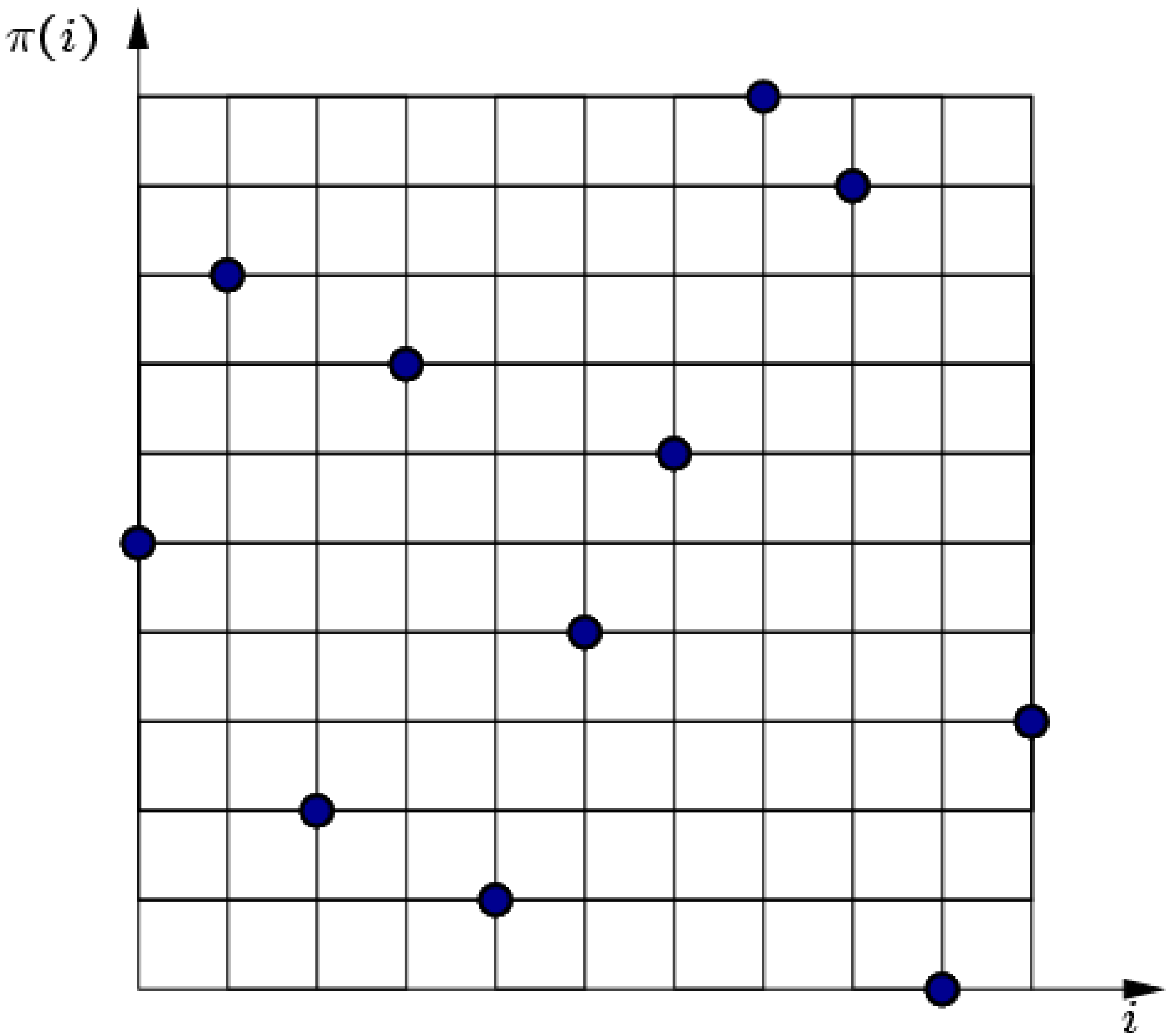
[Bóna 98]: pairs of patterns that are Wilf-equivalent to the “smooth” pair

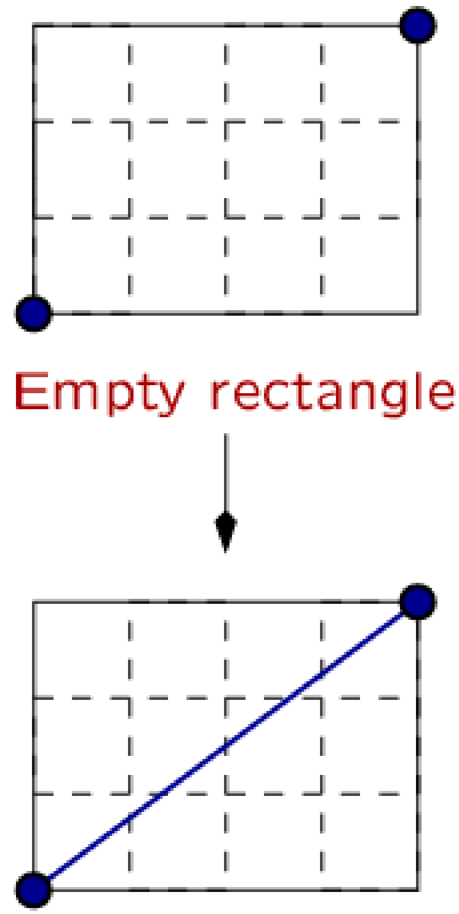
"Locally factorial" permutations

Locally factorial varieties generalize smooth varieties

Variety X_π	Smooth	Locally factorial
Patterns	1324 and 2143 [Lakshmibai-Sandhya 90]	?
Map L_π		Surjective [Woo-Yong 05]

A construction on permutations





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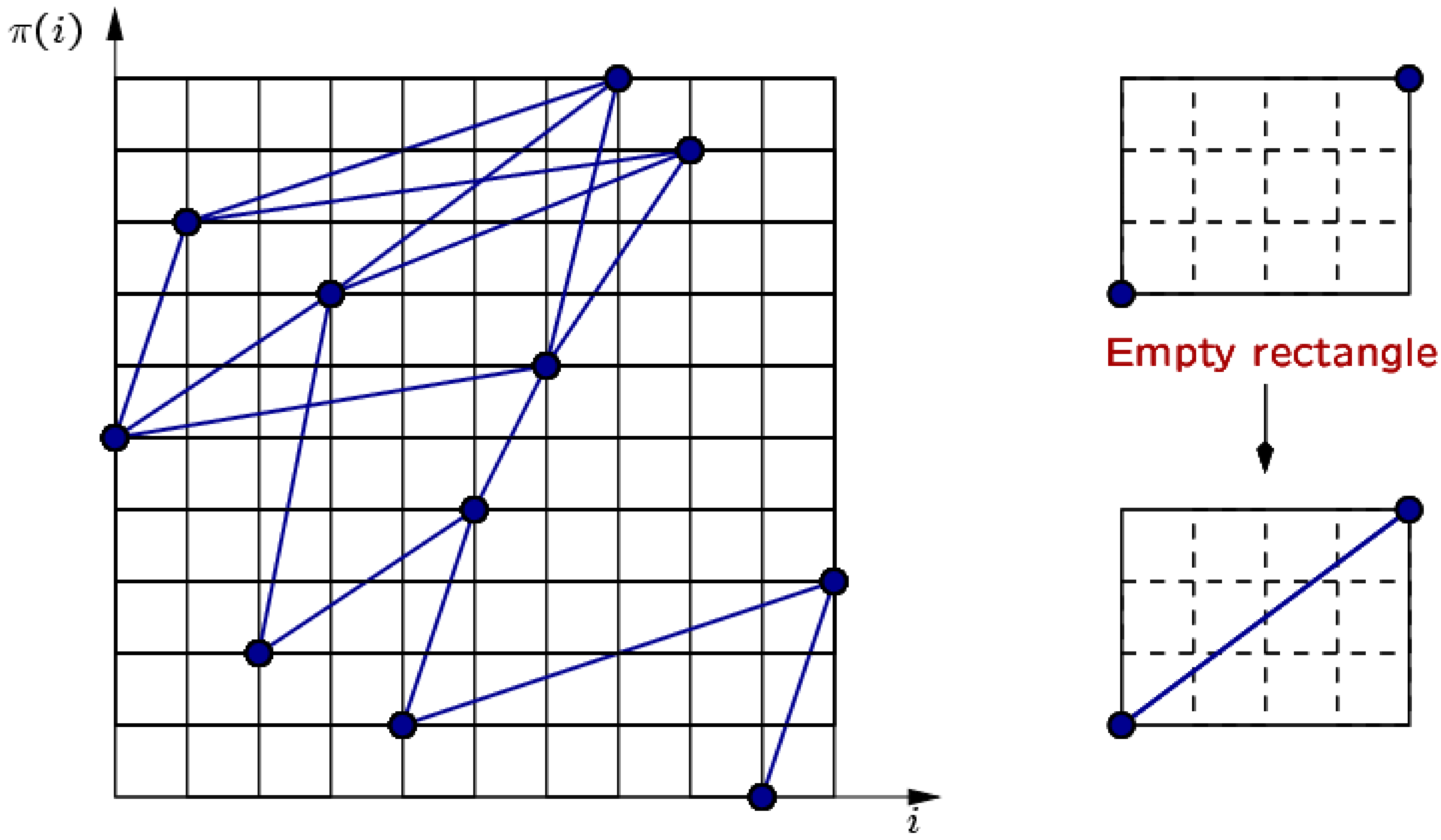
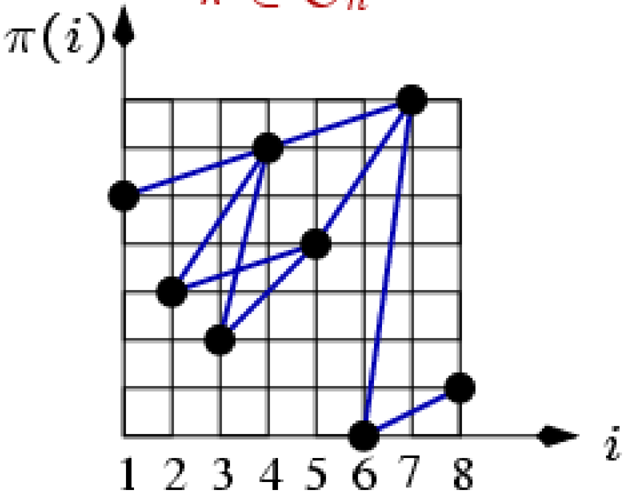


The Hasse diagram of the permutation, seen as a sub-poset of $\mathbb{N}^2$

Three related objects

$\pi \in \mathfrak{S}_n$



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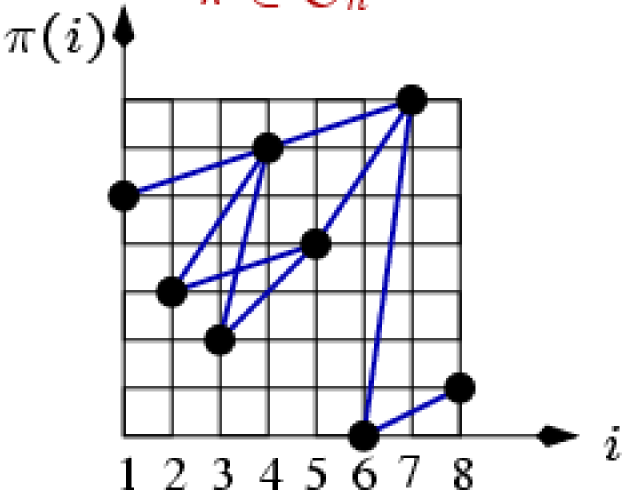
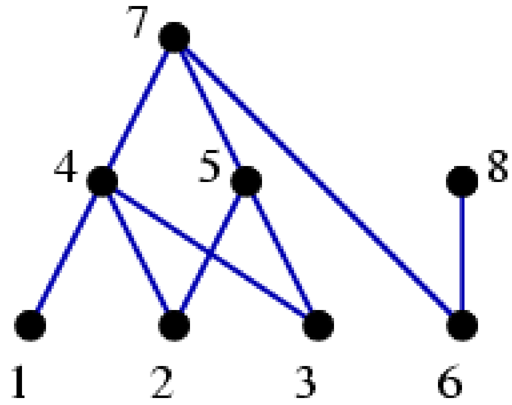


The graph G_π



$e(\pi)$ edges

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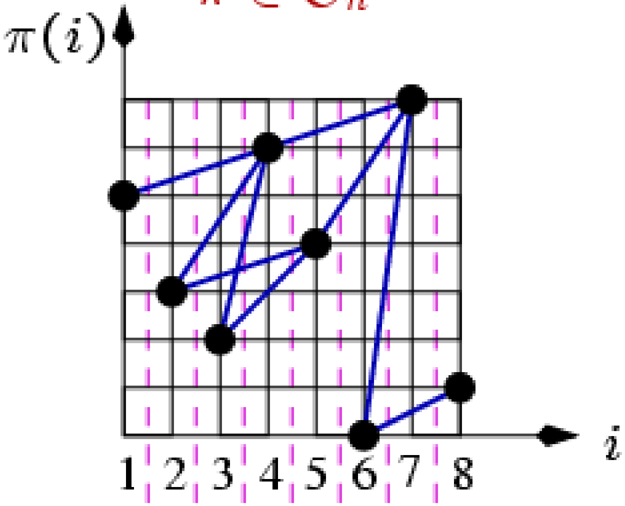

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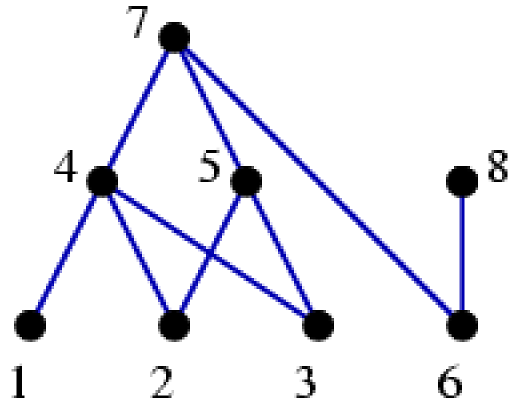

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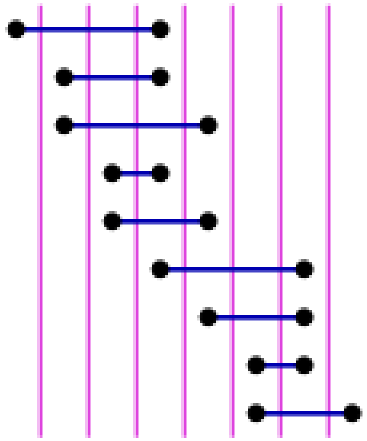
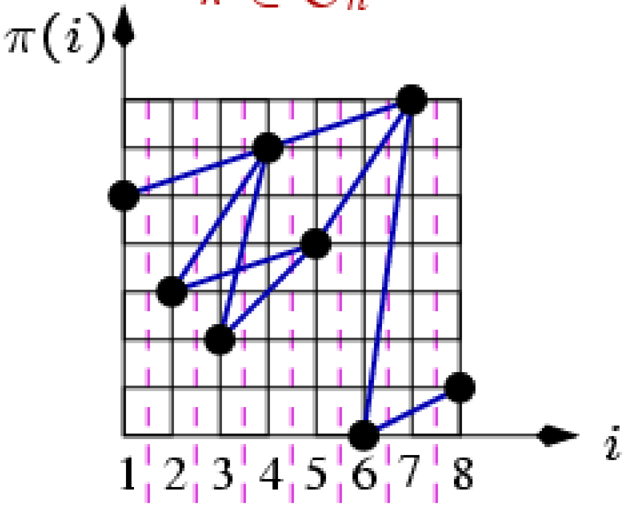


A collection of bars

Three related objects

$\pi \in \mathfrak{S}_n$



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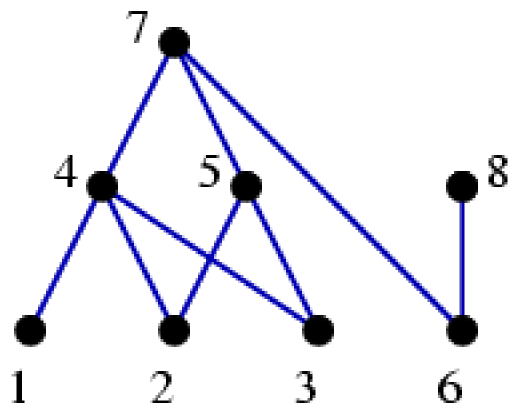

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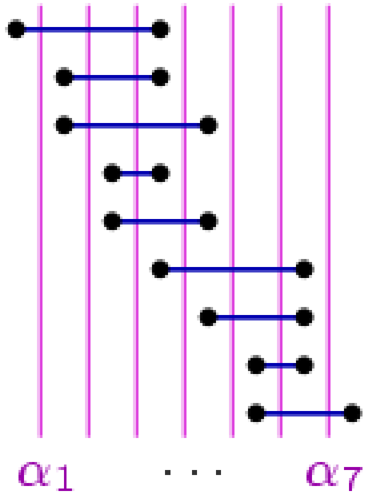

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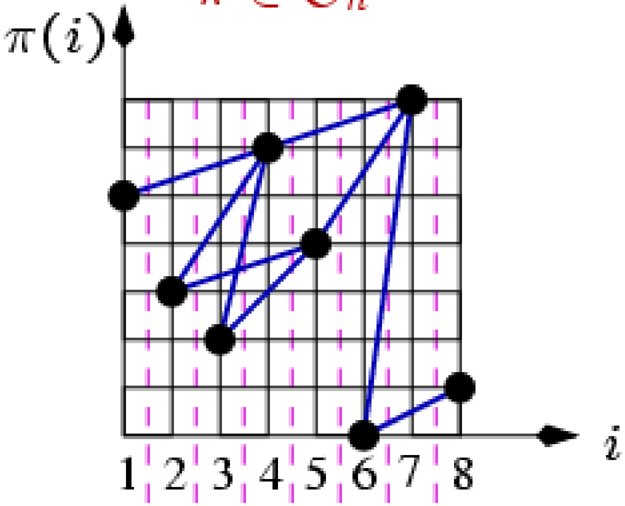

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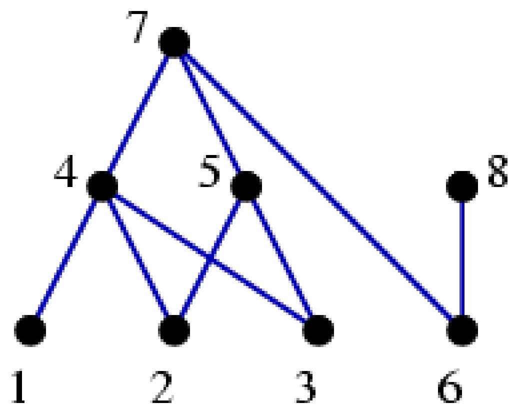

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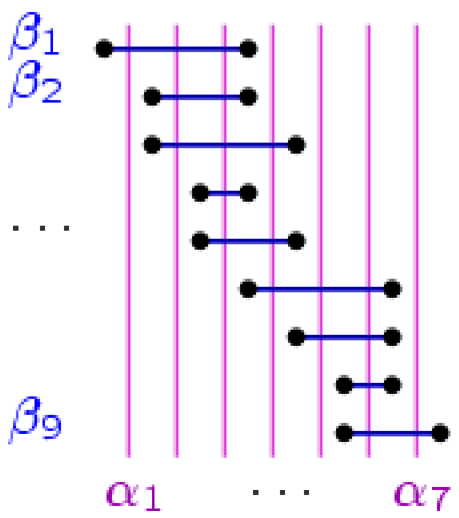


A collection of bars

$$L_\pi : \quad \mathbb{Z}^{n-1} \rightarrow \mathbb{Z}^{e(\pi)}$$

$$(\alpha_1, \ldots, \alpha_{n-1}) \mapsto (\beta_1, \ldots, \beta_{e(\pi)})$$

β_k: sum of the α_j's crossed by the kth bar

$(\beta_1 = \alpha_1 + \alpha_2 + \alpha_3,\ \beta_2 = \alpha_2 + \alpha_3,\ \cdots)$

The map L_π

“Locally factorial” permutations

Variety X_π	Smooth	Locally factorial
Patterns	1324 and 2143 [Lakshmibai-Sandhya 90]	
Map L_π		Surjective (onto) [Woo-Yong 05]

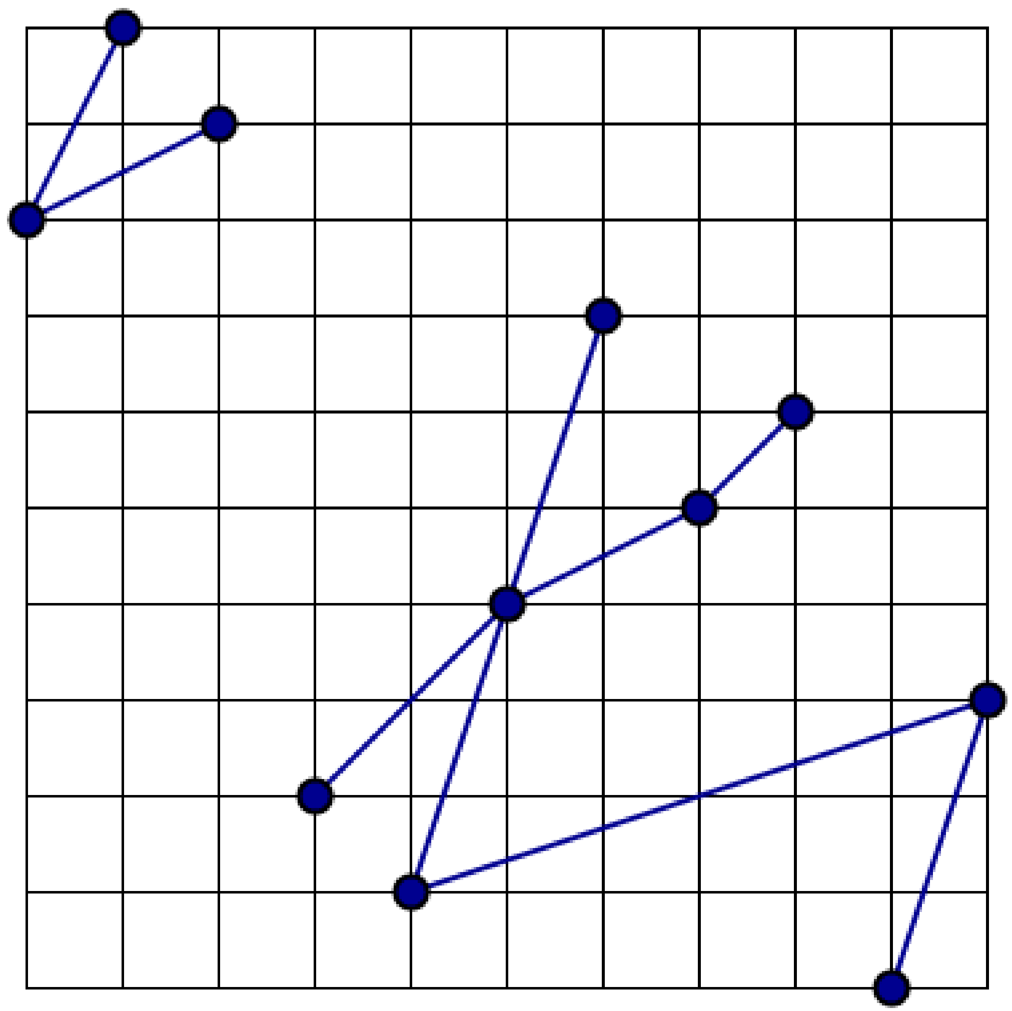
Part II. Characterization of “locally factorial” permutations

Thm 1: Characterization of "locally factorial" permutations

Variety X_π	Smooth	Locally factorial
Patterns	1324 and 2143 [Lakshmibai-Sandhya 90]	1324 and $21\bar{3}54$ [mbm-Butler 06]
Map L_π		Surjective [Woo-Yong 05]
Graph G_π		Forest (no cycle) [mbm-Butler 06]

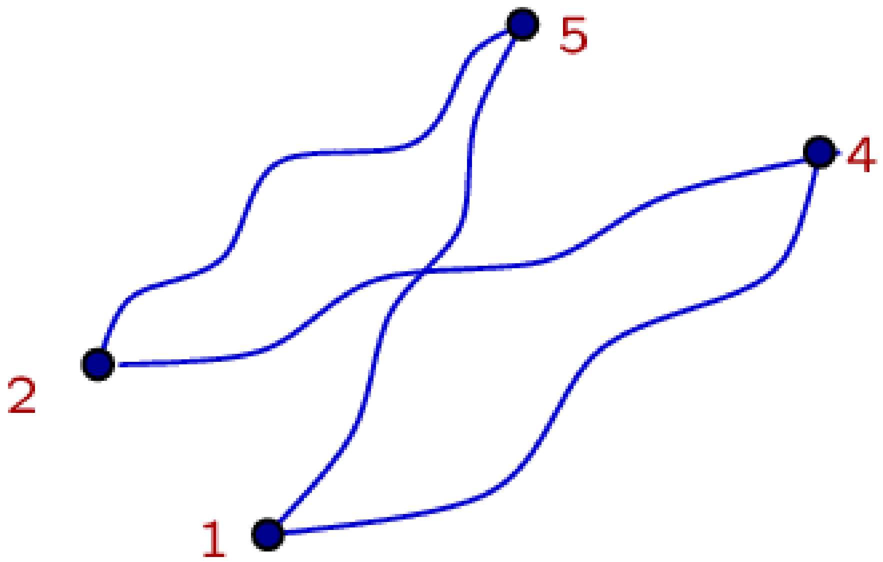
Terminology: "locally factorial" $\equiv$ forest-like

Forest-like permutations: an example

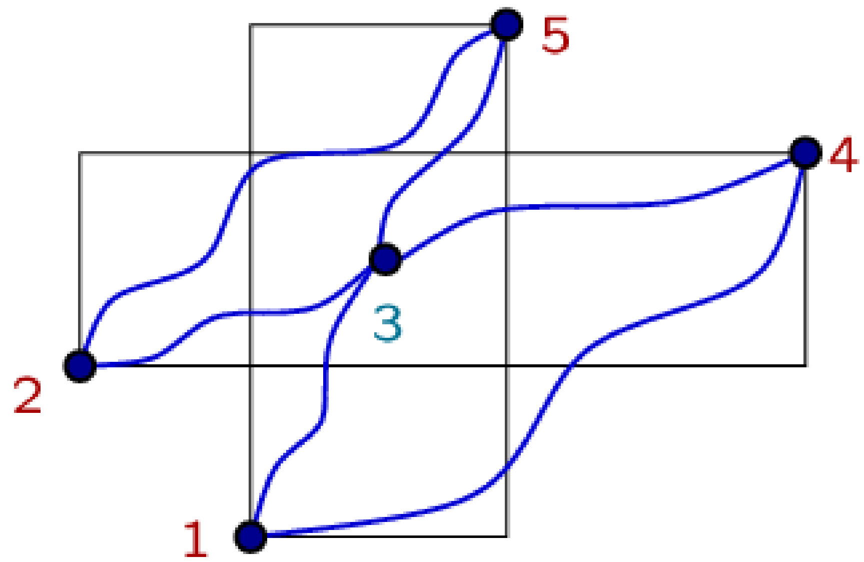


The pattern $21\bar{3}54$

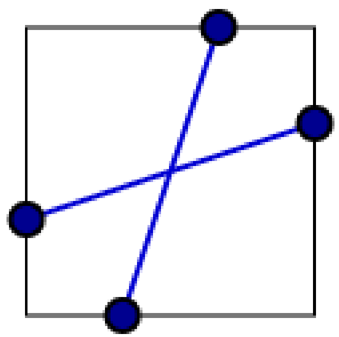
Def. The permutation π avoids $21\bar{3}54$ if every occurrence of 2154 (that is, of 2143) is a sub-occurrence of 21354.



$\Rightarrow$

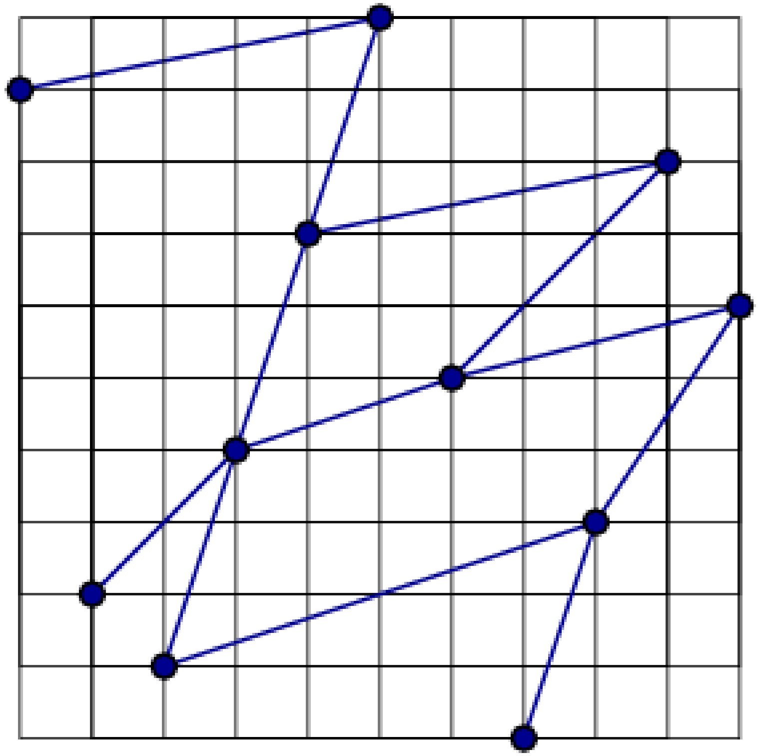


$\Rightarrow$ The permutation π avoids $21\bar{3}54$ iff the edges in the Hasse diagram of π do not cross: no occurrence of

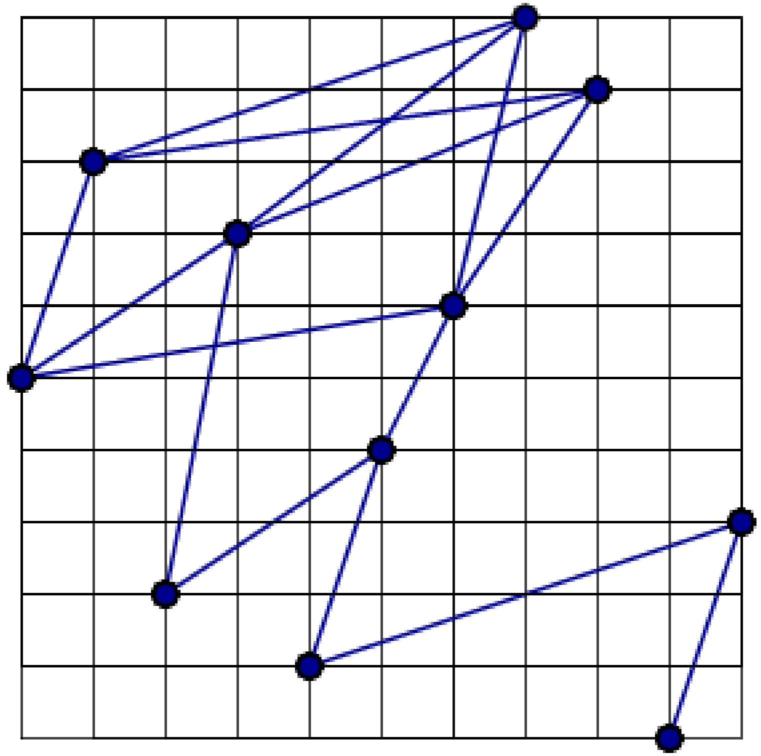


The pattern $21\bar{3}54$

The permutation π avoids $21\bar{3}54$ iff the associated Hasse diagram is planar.



Planar



Non-planar

Thm 1: Characterization of "locally factorial" permutations

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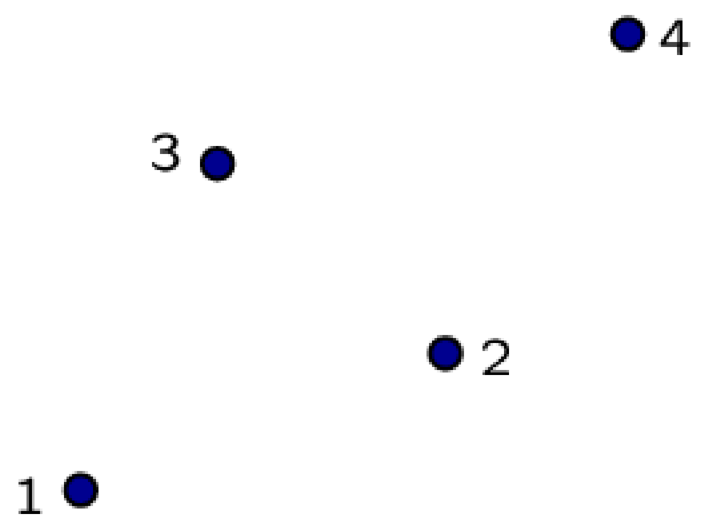
Patterns	1324 and 2143 [Lakshmibai-Sandhya 90]	1324 and $21\bar{3}54$ [mbm-Butler 06]
Map L_π		Surjective [Woo-Yong 05]
Graph G_π		Forest (no cycle) [mbm-Butler 06]

Terminology: "locally factorial" $\equiv$ forest-like

Why 1324 creates cycles in G_π and prevents surjectivity



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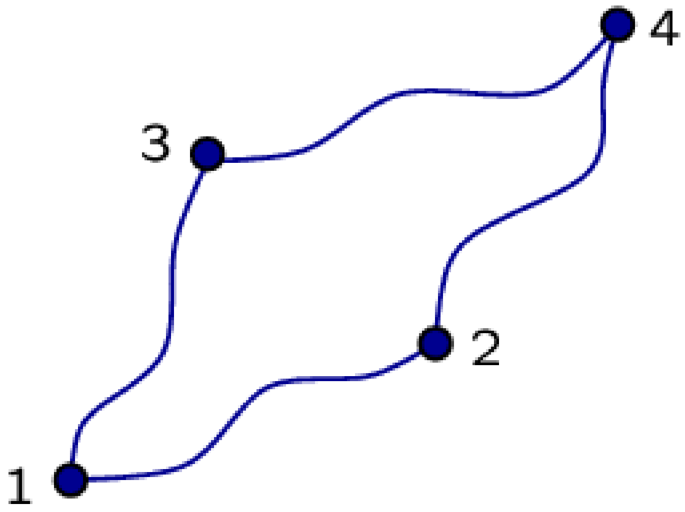

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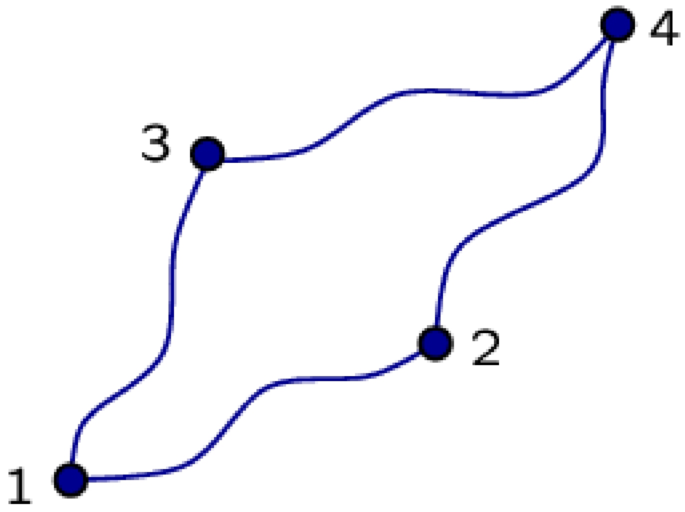

$\Downarrow$

Cycle

G_π is not a forest

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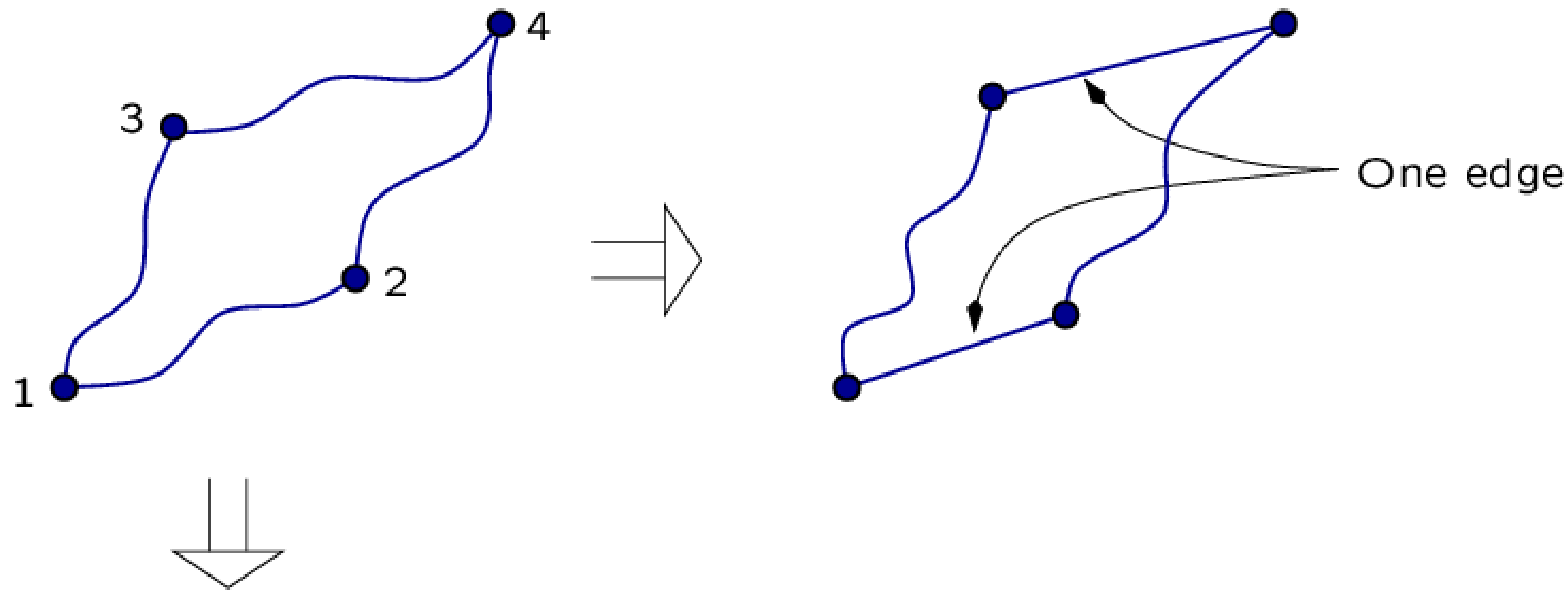

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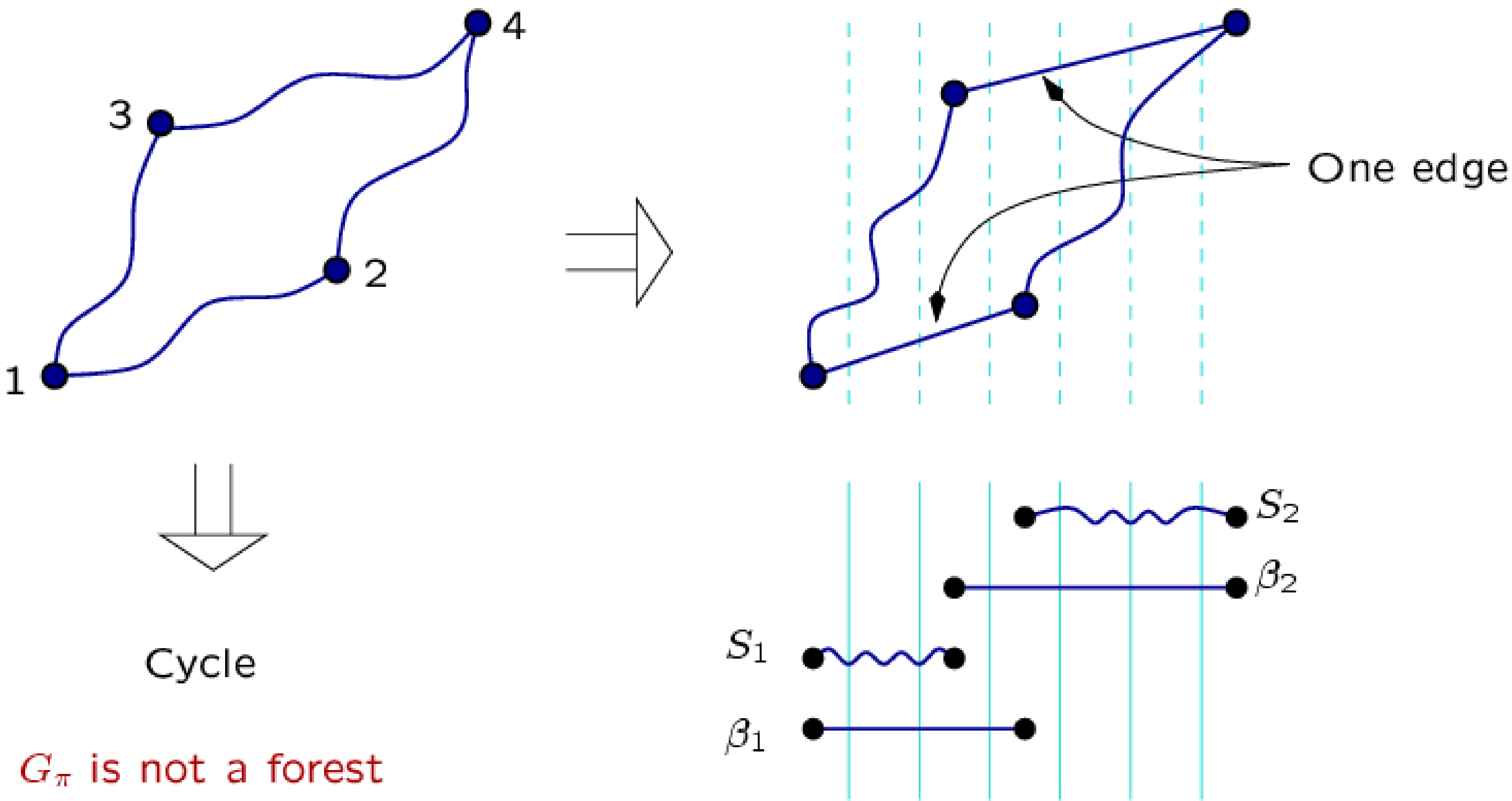

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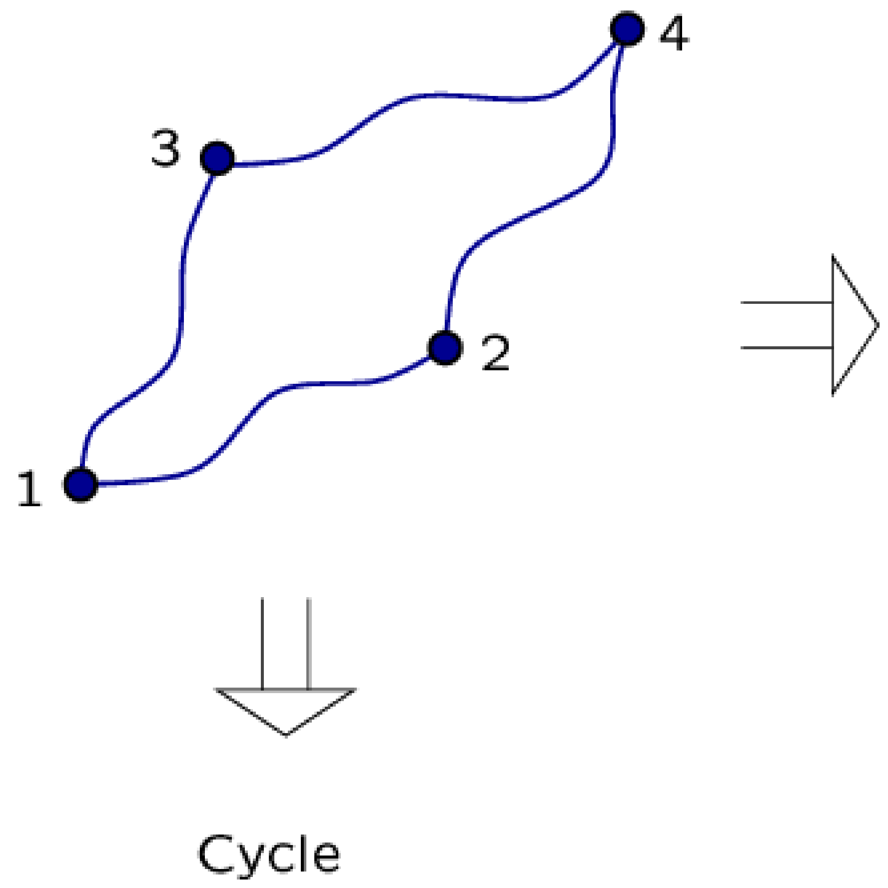
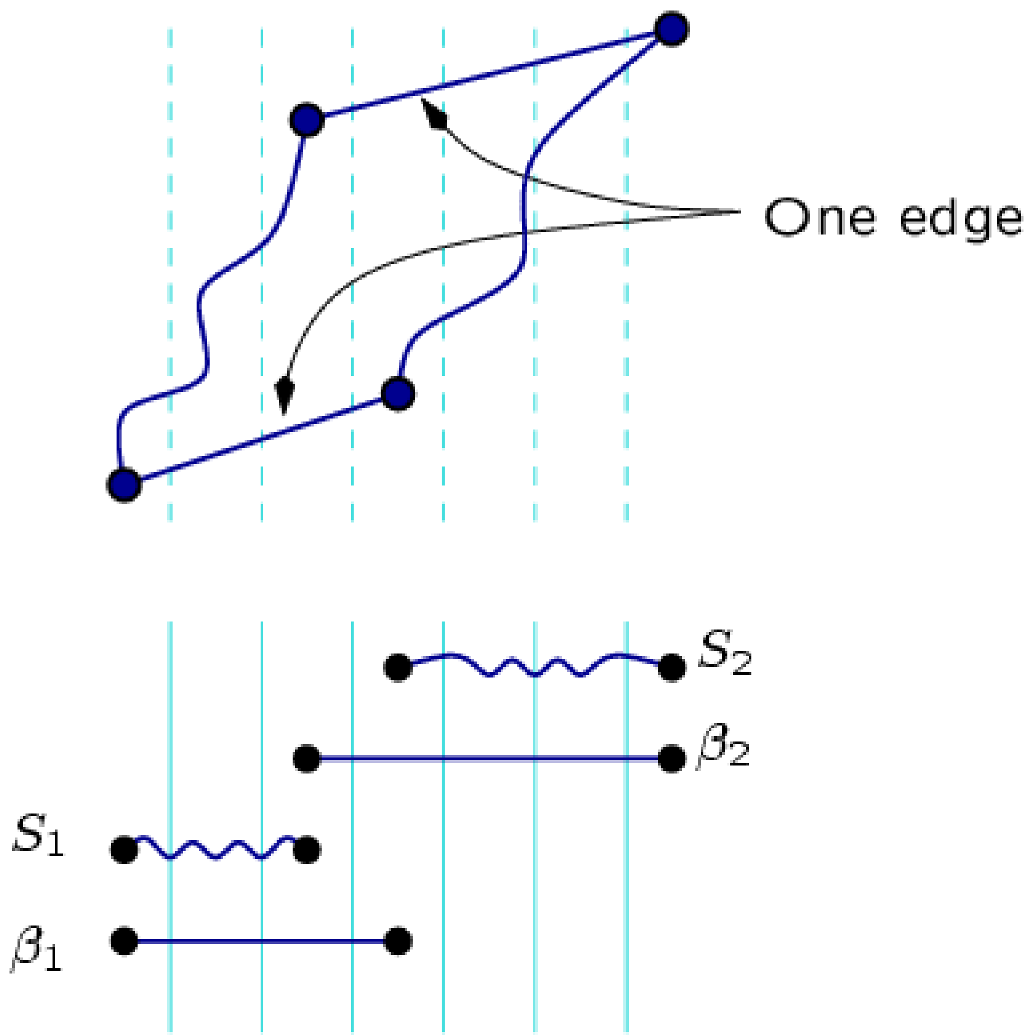


G_π is not a forest



$$\beta_1 + S_2 = \beta_2 + S_1$$

Linear dependence among the β_j's

$\Rightarrow L_\pi$ is not surjective

Part III. Enumeration of “locally factorial” (forest-like) permutations

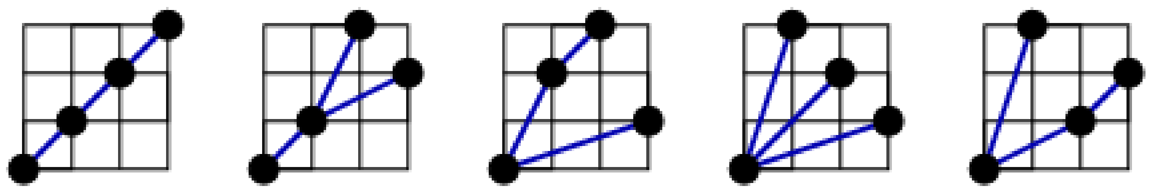
An easy case: *rooted* forest-like permutations

Def: π is rooted is $\pi(1) = 1$.

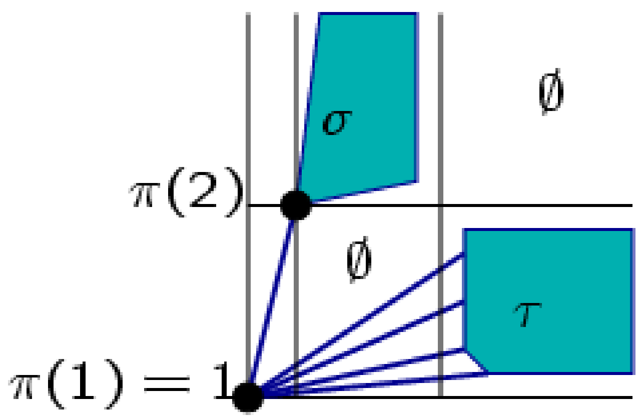
Lemma: A rooted permutation is forest-like iff it avoids 213. Hence $r_n = C_{n-1}$.

Proof. Must avoid 1324 (hence 213) and $21\bar{3}54$ (which holds automatically once 213 is forbidden). □

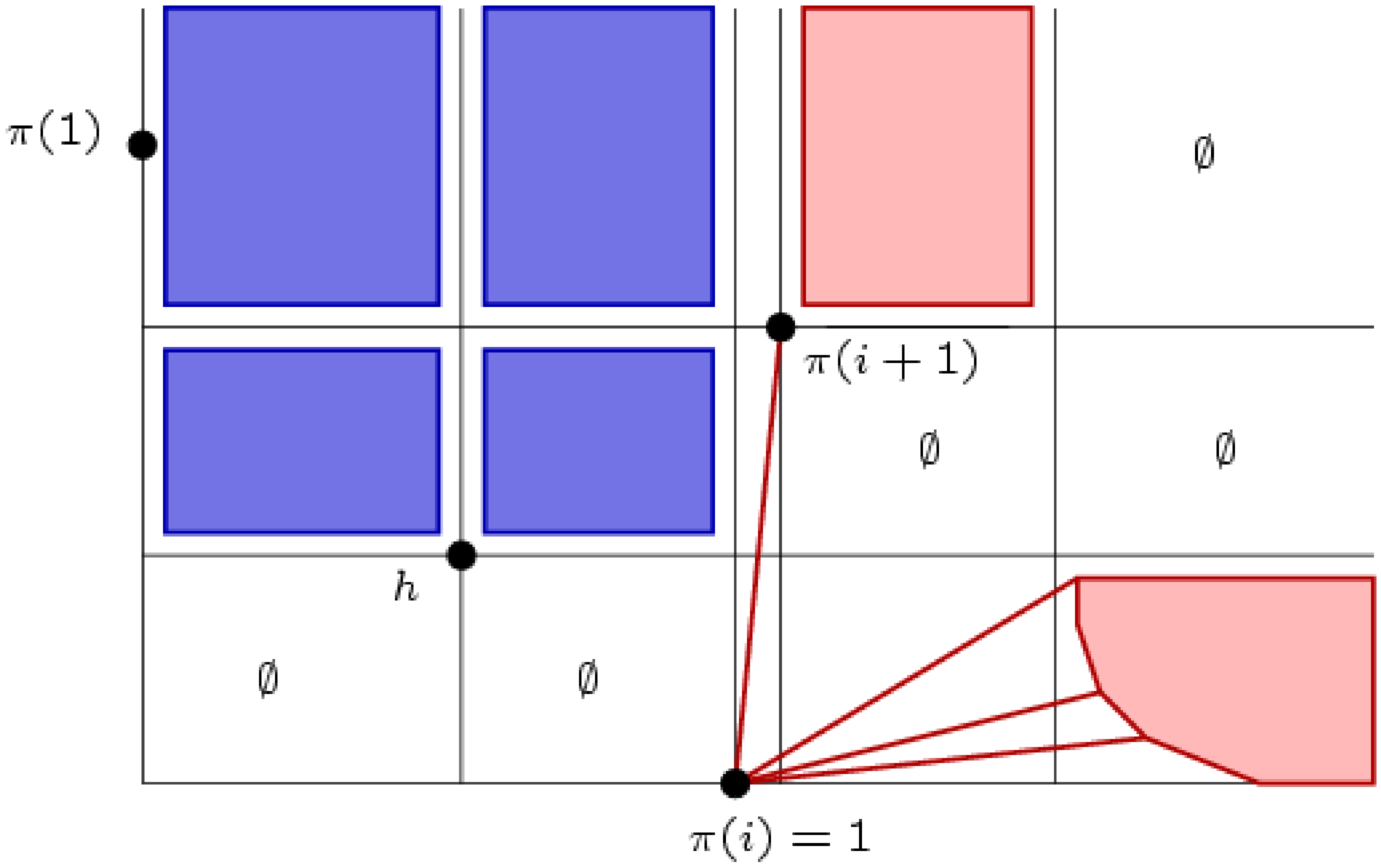
- Rooted forest-like permutations are in bijection with (rooted) plane trees



Recursive structure:



General case: structure of forest-like permutations



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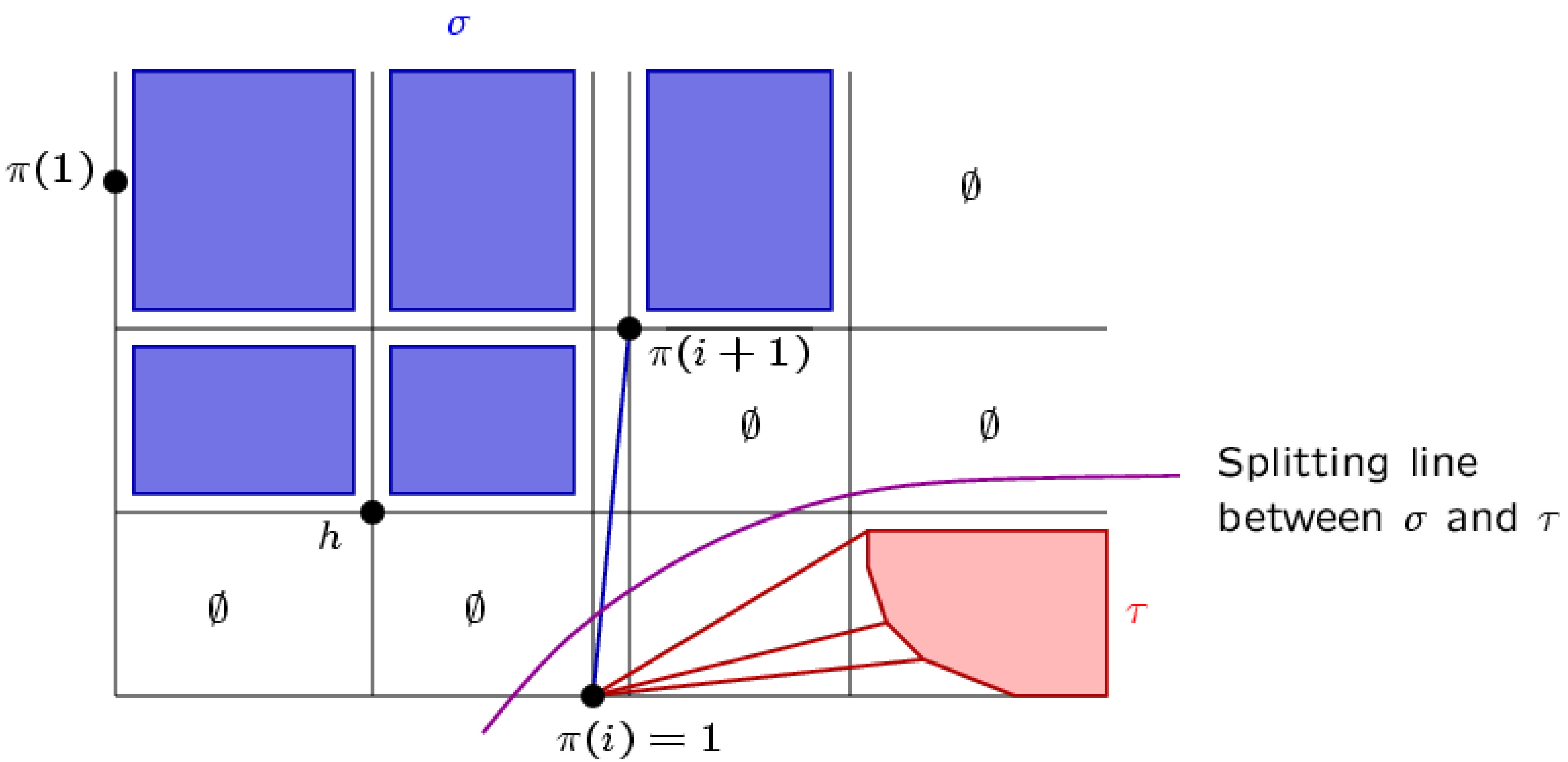


- If $\pi \mapsto (\sigma, \tau)$, then σ is forest-like, τ is rooted forest-like.
- To reconstruct π, one inserts 1 just before a right-to-left minimum of σ.

General case: functional equations

- Let $\mathcal{F}(u) \equiv \mathcal{F}(x,u)$ be the generating function of forest-like permutations, counted by the size (x) and number of r-l minima (u).

- Define similarly $\mathcal{R}(u) \equiv \mathcal{R}(x,u)$ for *rooted* forest-like permutations. Then

$$\begin{aligned}
\mathcal{F}(u) &= xu + xu\mathcal{F}(1) + xu^2\frac{\mathcal{F}(u)-\mathcal{F}(1)}{u-1} + (\mathcal{R}(u)-xu)\mathcal{F}'(1),\\
\mathcal{R}(u) &= xu + xu\mathcal{R}(u) + (\mathcal{R}(u)-xu)\mathcal{R}(1),
\end{aligned}$$

where $\mathcal{F}'(1) = \frac{\partial \mathcal{F}}{\partial u}(x,1)$.

Solution via the kernel method

- The number of rooted forest-like permutations of $\mathfrak{S}_n$ is the Catalan number C_{n-1}:

$$r_n = \frac{1}{n}\binom{2n-2}{n-1} = C_{n-1}.$$

The associated generating function is

$$R(x) = \sum_{n\geq 1} r_n x^n = \frac{1-\sqrt{1-4x}}{2}.$$

- The generating function of forest-like permutations is

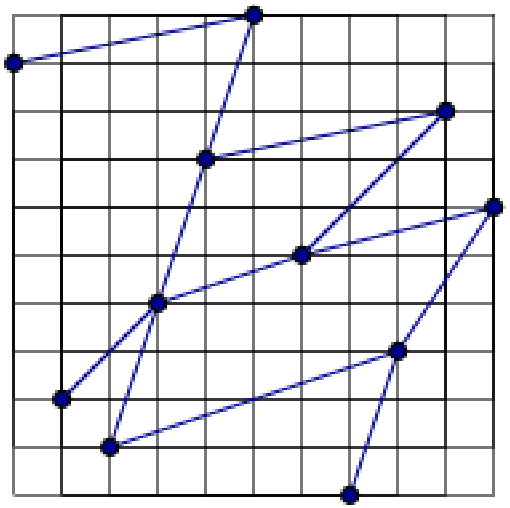
$$F(x) = \sum_{n\geq 1} f_n x^n = \frac{(1-x)(1-4x+2x^2)-(1-5\,x)\sqrt{1-4\,x}}{2(1-5\,x+2\,x^2-x^3)}.$$

Part IV. Questions

Questions

1. How many planar permutations are there? ($21\bar{3}54$-avoiding)

1, 2, 6, 23, 104, 530, 2958, 17734, 112657...



2. Count permutations by the size and the number of edges in their Hasse diagram (= number of elements covering them in the Bruhat order)

3. "Gorenstein" permutations

"Gorenstein" permutations

Variety X_π	Smooth	Locally factorial	Gorenstein
Patterns	1324 and 2143 [Lakshmibai-Sandhya 90]	1324 and $21\bar{3}54$ [mbm-Butler 06]	?
Map L_π		Surjective [Woo-Yong 05]	Reaches $(1, 1, \ldots, 1)$ [Woo-Yong 05]
Graph G_π		Forest [mbm-Butler 06]	?

That’s it!