On some properties of permutation tableaux

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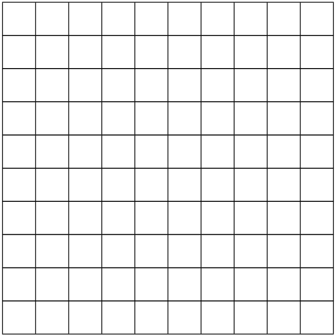
(column) Every column contains at least one 1.

Example

$n = 20, \ k = 10$



Example

$n = 20, \ k = 10$

									2
									2
									2
							2	2	2
							2	2	2
						2	2	2	2
					2	2	2	2	2

Example

$n = 20, \ k = 10$

1	1								
					1	1			
1		1					1		1
						1		1	2
				1					2
		1							2
					1		2	2	2
					1		2	2	2
1						2	2	2	2
	1		1		2	2	2	2	2

Example

$n = 20, \ k = 10$

1	1								
					1	1			
1	1	1			1	1	1		1
						1	1	1	2
				1	1	1	1	1	2
		1		1	1	1	1	1	2
					1	1	2	2	2
					1	1	2	2	2
1	1	1		1	1	2	2	2	2
	1	1	1	1	2	2	2	2	2

Example

$n = 20, \ k = 10$

1	1								
					1	1			
1	1	1			1	1	1		1
						1	1	1	2
				1	1	1	1	1	2
		1		1	1	1	1	1	2
					1	1	2	2	2
					1	1	2	2	2
1	1	1		1	1	2	2	2	2
	1	1	1	1	2	2	2	2	2

Example

$n = 20, \ k = 10$

1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	1	1	0	0	0
1	1	1	0	0	1	1	1	0	1
0	0	0	0	0	0	1	1	1	2
0	0	0	0	1	1	1	1	1	2
0	0	1	0	1	1	1	1	1	2
0	0	0	0	0	1	1	2	2	2
0	0	0	0	0	1	1	2	2	2
1	1	1	0	1	1	2	2	2	2
0	1	1	1	1	2	2	2	2	2

Example

$n = 20,\ k = 10$

1	1	0	0	0	0	0	0	0	0
0	0	0	0	0	1	1	0	0	0
1	1	1	0	0	1	1	1	0	1
0	0	0	0	0	0	1	1	1	
0	0	0	0	1	1	1	1	1	
0	0	1	0	1	1	1	1	1	
0	0	0	0	0	1	1			
0	0	0	0	0	1	1			
1	1	1	0	1	1				
0	1	1	1	1					

Example

$n = 20, \; k = 10$

1	1								
					1	1			
1	1	1			1	1	1		1
						1	1	1	
				1	1	1	1	1	
		1		1	1	1	1	1	
					1	1			
					1	1			
1	1	1		1	1				
	1	1	1	1					

Example

$n = 20, \ k = 10$

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Boundary Path

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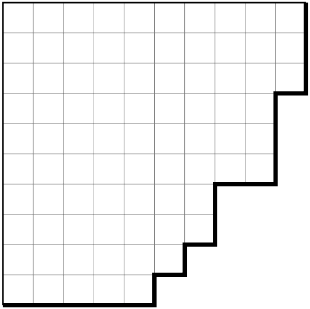
- Label the path steps 1 through n from NE to SW end.
- Label the rows and columns with the label of the corresponding step.
- Label each cell p with its row and column labels $(l_r(p), l_c(p))$.
- Note that $l_c(p) > l_r(p)$ for every cell p in the tableau.

Example



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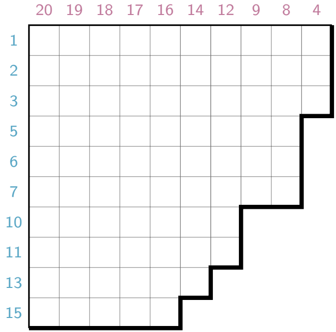

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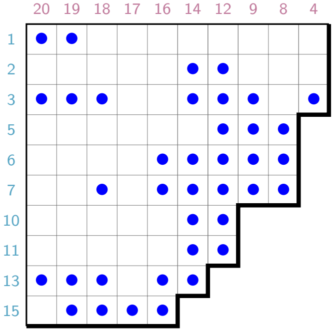

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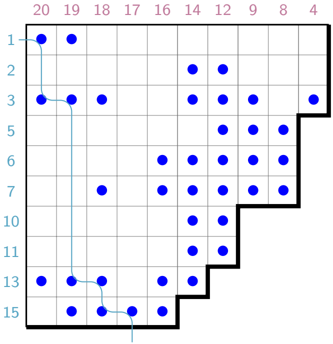

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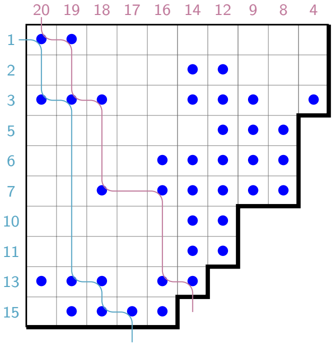

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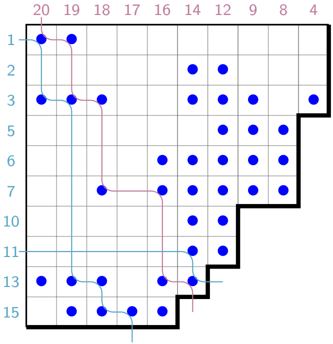

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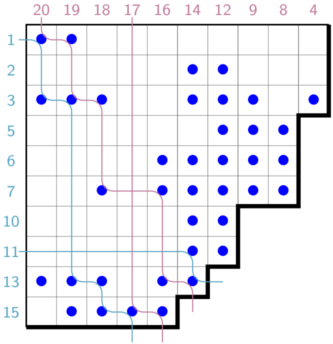

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$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 17 & 7 & 18 & 3 & 8 & 11 & 15 & 5 & 4 & 12 & 13 & 2 & 20 & 6 & 19 & 10 & 16 & 9 & 1 & 14 \end{pmatrix}$$

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- Reconstruction by rows from bottom to top is similar.
- We will reconstruct T from π by columns from left to right (and by rows from top to bottom).

Leftmost column

- We want to find the row labels of 1s in the leftmost column.

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- Then $\pi(n) < \pi(i_1) < \pi(i_2) < \cdots < \pi(i_k) = n$ ...

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- ...and for $j \in (i_r, i_{r+1})$, $\pi(j) < \pi(i_r)$.
- Hence, $\pi(i_1), \pi(i_2), \ldots, \pi(i_k) = n$ are the LR-minima of the subsequence of π on letters greater than $\pi(n)$.
- If T' is T without the leftmost column, then $\Phi(T) = \Phi(T')(i_1 \ i_2 \ \ldots \ i_k \ n)$.

Column Decomposition

- The column decomposition of a permutation π is the (unique) representation of π as a product $c_{n-k}c_{n-k-1}\ldots c_1$ of increasing cycles c_i $(1 \leq i \leq n-k)$ such that maximal elements of c_i's are distinct from one another and from other elements in c_i's, and if

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    - $i < j$,
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  then c_j also contains b.
- This condition is equivalent to the 1-hinge rule for tableaux.

Example of Column Decomposition

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
17	7	18	3	8	11	15	5	4	12	13	2	20	6	19	10	16	9	1	14
14	7	17	3	8	11	15	5	4	12	13	2	18	6	19	10	16	9	1	20
1	7	14	3	8	11	15	5	4	12	13	2	17	6	18	10	16	9	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	17	10	16	18	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	16	10	17	18	19	20
1	7	9	3	8	10	11	5	4	12	13	2	14	6	15	16	17	18	19	20
1	6	7	3	8	9	10	5	4	11	12	2	13	14	15	16	17	18	19	20
1	2	6	3	7	8	9	5	4	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	6	7	8	5	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

$\pi =$

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14	7	17	3	8	11	15	5	4	12	13	2	18	6	19	10	16	9	1	20
1	7	14	3	8	11	15	5	4	12	13	2	17	6	18	10	16	9	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	17	10	16	18	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	16	10	17	18	19	20
1	7	9	3	8	10	11	5	4	12	13	2	14	6	15	16	17	18	19	20
1	6	7	3	8	9	10	5	4	11	12	2	13	14	15	16	17	18	19	20
1	2	6	3	7	8	9	5	4	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	6	7	8	5	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
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17	7	18	3	8	11	15	5	4	12	13	2	20	6	19	10	16	9	1	14
14	7	17	3	8	11	15	5	4	12	13	2	18	6	19	10	16	9	1	20
1	7	14	3	8	11	15	5	4	12	13	2	17	6	18	10	16	9	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	17	10	16	18	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	16	10	17	18	19	20
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(1 3 13 20)

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1	2	4	3	6	7	8	5	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

$\pi =$

(1 3 13 15 19)

(1 3 13 20)

Example of Column Decomposition

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
17	7	18	3	8	11	15	5	4	12	13	2	20	6	19	10	16	9	1	14
14	7	17	3	8	11	15	5	4	12	13	2	18	6	19	10	16	9	1	20
1	7	14	3	8	11	15	5	4	12	13	2	17	6	18	10	16	9	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	17	10	16	18	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	16	10	17	18	19	20
1	7	9	3	8	10	11	5	4	12	13	2	14	6	15	16	17	18	19	20
1	6	7	3	8	9	10	5	4	11	12	2	13	14	15	16	17	18	19	20
1	2	6	3	7	8	9	5	4	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	6	7	8	5	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

$\pi =$

(3 7 13 15 18)(1 3 13 15 19)

(1 3 13 20)

Example of Column Decomposition

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
17	7	18	3	8	11	15	5	4	12	13	2	20	6	19	10	16	9	1	14
14	7	17	3	8	11	15	5	4	12	13	2	18	6	19	10	16	9	1	20
1	7	14	3	8	11	15	5	4	12	13	2	17	6	18	10	16	9	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	17	10	16	18	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	16	10	17	18	19	20
1	7	9	3	8	10	11	5	4	12	13	2	14	6	15	16	17	18	19	20
1	6	7	3	8	9	10	5	4	11	12	2	13	14	15	16	17	18	19	20
1	2	6	3	7	8	9	5	4	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	6	7	8	5	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

$\pi =$

(15 17)(3 7 13 15 18)(1 3 13 15 19)

(1 3 13 20)

Example of Column Decomposition

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
17	7	18	3	8	11	15	5	4	12	13	2	20	6	19	10	16	9	1	14
14	7	17	3	8	11	15	5	4	12	13	2	18	6	19	10	16	9	1	20
1	7	14	3	8	11	15	5	4	12	13	2	17	6	18	10	16	9	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	17	10	16	18	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	16	10	17	18	19	20
1	7	9	3	8	10	11	5	4	12	13	2	14	6	15	16	17	18	19	20
1	6	7	3	8	9	10	5	4	11	12	2	13	14	15	16	17	18	19	20
1	2	6	3	7	8	9	5	4	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	6	7	8	5	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

$\pi =$

(6 7 13 15 16)(15 17)(3 7 13 15 18)(1 3 13 15 19)

(1 3 13 20)

Example of Column Decomposition

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
17	7	18	3	8	11	15	5	4	12	13	2	20	6	19	10	16	9	1	14
14	7	17	3	8	11	15	5	4	12	13	2	18	6	19	10	16	9	1	20
1	7	14	3	8	11	15	5	4	12	13	2	17	6	18	10	16	9	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	17	10	16	18	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	16	10	17	18	19	20
1	7	9	3	8	10	11	5	4	12	13	2	14	6	15	16	17	18	19	20
1	6	7	3	8	9	10	5	4	11	12	2	13	14	15	16	17	18	19	20
1	2	6	3	7	8	9	5	4	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	6	7	8	5	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

$\pi =$

(2 3 6 7 10 11 13 14)(6 7 13 15 16)(15 17)(3 7 13 15 18)(1 3 13 15 19)

(1 3 13 20)

Example of Column Decomposition

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
17	7	18	3	8	11	15	5	4	12	13	2	20	6	19	10	16	9	1	14
14	7	17	3	8	11	15	5	4	12	13	2	18	6	19	10	16	9	1	20
1	7	14	3	8	11	15	5	4	12	13	2	17	6	18	10	16	9	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	17	10	16	18	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	16	10	17	18	19	20
1	7	9	3	8	10	11	5	4	12	13	2	14	6	15	16	17	18	19	20
1	6	7	3	8	9	10	5	4	11	12	2	13	14	15	16	17	18	19	20
1	2	6	3	7	8	9	5	4	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	6	7	8	5	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

$\pi =$ (2 3 5 6 7 10 11 12)(2 3 6 7 10 11 13 14)(6 7 13 15 16)(15 17)(3 7 13 15 18)(1 3 13 15 19)(1 3 13 20)

Example of Column Decomposition

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
17	7	18	3	8	11	15	5	4	12	13	2	20	6	19	10	16	9	1	14
14	7	17	3	8	11	15	5	4	12	13	2	18	6	19	10	16	9	1	20
1	7	14	3	8	11	15	5	4	12	13	2	17	6	18	10	16	9	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	17	10	16	18	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	16	10	17	18	19	20
1	7	9	3	8	10	11	5	4	12	13	2	14	6	15	16	17	18	19	20
1	6	7	3	8	9	10	5	4	11	12	2	13	14	15	16	17	18	19	20
1	2	6	3	7	8	9	5	4	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	6	7	8	5	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

$\pi =$

$(3\,5\,6\,7\,9)(2\,3\,5\,6\,7\,10\,11\,12)(2\,3\,6\,7\,10\,11\,13\,14)(6\,7\,13\,15\,16)(15\,17)(3\,7\,13\,15\,18)(1\,3\,13\,15\,19)(1\,3\,13\,20)$

Example of Column Decomposition

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
17	7	18	3	8	11	15	5	4	12	13	2	20	6	19	10	16	9	1	14
14	7	17	3	8	11	15	5	4	12	13	2	18	6	19	10	16	9	1	20
1	7	14	3	8	11	15	5	4	12	13	2	17	6	18	10	16	9	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	17	10	16	18	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	16	10	17	18	19	20
1	7	9	3	8	10	11	5	4	12	13	2	14	6	15	16	17	18	19	20
1	6	7	3	8	9	10	5	4	11	12	2	13	14	15	16	17	18	19	20
1	2	6	3	7	8	9	5	4	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	6	7	8	5	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

$\pi =$ (5 6 7 8)(3 5 6 7 9)(2 3 5 6 7 10 11 12)(2 3 6 7 10 11 13 14)(6 7 13 15 16)(15 17)(3 7 13 15 18)(1 3 13 15 19)(1 3 13 20)

Example of Column Decomposition

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
17	7	18	3	8	11	15	5	4	12	13	2	20	6	19	10	16	9	1	14
14	7	17	3	8	11	15	5	4	12	13	2	18	6	19	10	16	9	1	20
1	7	14	3	8	11	15	5	4	12	13	2	17	6	18	10	16	9	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	17	10	16	18	19	20
1	7	9	3	8	11	14	5	4	12	13	2	15	6	16	10	17	18	19	20
1	7	9	3	8	10	11	5	4	12	13	2	14	6	15	16	17	18	19	20
1	6	7	3	8	9	10	5	4	11	12	2	13	14	15	16	17	18	19	20
1	2	6	3	7	8	9	5	4	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	6	7	8	5	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	4	3	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

$\pi =$
(3 4)(5 6 7 8)(3 5 6 7 9)(2 3 5 6 7 10 11 12)(2 3 6 7 10 11 13 14)(6 7 13 15 16)(15 17)(3 7 13 15 18)(1 3 13 15 19)
(1 3 13 20)

Example of Column Decomposition

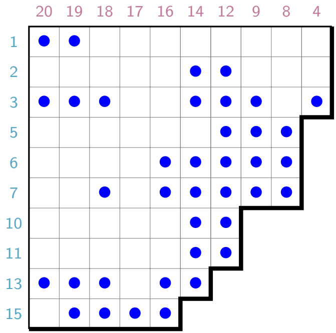
$\pi =$

$(3\ 4)(5\ 6\ 7\ 8)(3\ 5\ 6\ 7\ 9)(2\ 3\ 5\ 6\ 7\ 10\ 11\ 12)(2\ 3\ 6\ 7\ 10\ 11\ 13\ 14)(6\ 7\ 13\ 15\ 16)(15\ 17)(3\ 7\ 13\ 15\ 18)(1\ 3\ 13\ 15\ 19)(1\ 3\ 13\ 20)$

Example of Column Decomposition

$\pi =$

(3 4)(5 6 7 8)(3 5 6 7 9)(2 3 5 6 7 10 11 12)(2 3 6 7 10 11 13 14)(6 7 13 15 16)(15 17)(3 7 13 15 18)(1 3 13 15 19)(1 3 13 20)



Row Decomposition

- The row decomposition of a permutation π is the (unique) representation of π as a product $c_k c_{k-1} \dots c_1$ of decreasing cycles c_i $(1 \leq i \leq k)$ such that minimal elements of c_i's are distinct from one another and from other elements in c_i's, and if
  - $i < j$,
  - c_i contains b,
  - c_j contains $c > a$,
  - $c > b > a$,

  then c_j also contains b.
- This condition is also equivalent to the 1-hinge rule for tableaux.

Example of Row Decomposition

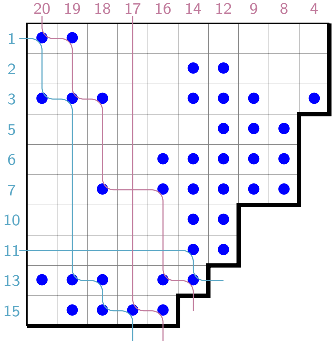
$\pi =$

(19 18 17 16 14 15)(20 19 18 16 14 13)(14 12 11)(14 12 10)(18 16 14 12 9 8 7)(16 14 12 9 8 6)(12 9 8 5)(20 19 18 14 12 9 4 3)

(14 12 2)(20 19 1)

	20	19	18	17	16	14	12	9	8	4
1	●	●								
2						●	●			
3	●	●	●			●	●	●		●
5							●	●	●	
6					●	●	●	●	●	
7			●		●	●	●	●	●	
10						●	●			
11						●	●			
13	●	●	●		●	●				
15		●	●	●	●					

Alignments and Crossings



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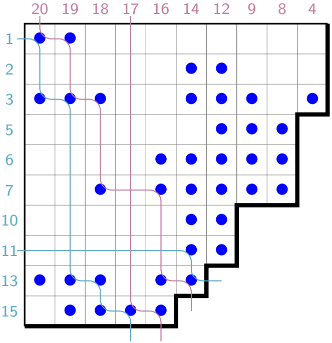

- Intersections: EE, NN, EN (E < N)

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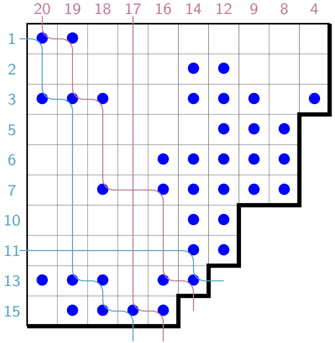

- Intersections: EE, NN, EN $(\mathrm{E} < \mathrm{N})$
- Non-intersections: NE $(\mathrm{N} < \mathrm{E})$

Alignments and Crossings

	20	19	18	17	16	14	12	9	8	4
1	●	●	NN	NN	NN	NN	NN	NN	NN	NN
2	EE	EE	EN	EN	EN	●	●	NN	NN	NN
3	●	●	●	NN	NN	●	●	●	NN	●
5	EE	EE	EN	EN	EN	EN	●	●	●	NE
6	EE	EE	EN	EN	●	●	●	●	●	NE
7	EE	EE	●	NN	●	●	●	●	●	NE
10	EE	EE	EE	EN	EN	●	●	NE	NE	NE
11	EE	EE	EE	EN	EN	●	●	NE	NE	NE
13	●	●	●	EN	●	●	NE	NE	NE	NE
15	EE	●	●	●	●	NE	NE	NE	NE	NE

- Intersections: EE, NN, EN $(\mathrm{E} < \mathrm{N})$
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Statistics on Permutation Tableaux

Let $\pi = \Phi(T)$. Define

$$A_{EE}(\pi) = |\{(i,j) \mid j < i \leq \pi(i) < \pi(j)\}| = \#\mathrm{EE}(T)$$
$$A_{NN}(\pi) = |\{(i,j) \mid \pi(j) < \pi(i) < i < j\}| = \#\mathrm{NN}(T)$$
$$A_{EN}(\pi) = |\{(i,j) \mid j \leq \pi(j) < \pi(i) < i\}| = \#\mathrm{EN}(T)$$
$$A_{NE}(\pi) = |\{(i,j) \mid \pi(i) < i < j \leq \pi(j)\}| = \#\mathrm{NE}(T) = \#2\mathrm{s}(T)$$
$$C_{EE}(\pi) = |\{(i,j) \mid j < i \leq \pi(j) < \pi(i)\}|$$
$$C_{NN}(\pi) = |\{(i,j) \mid \pi(i) < \pi(j) < i < j\}|$$

It can be shown that

$$C_{EE}(\pi) + C_{NN}(\pi) = \#\text{nontop } 1\text{s}(T)$$

Steingrímsson and Williams define a map $\Psi : \mathfrak{S}_n \to \mathfrak{S}_n$ that takes descents (des) and ascents (ndes) to weak excedances (wex) and deficiencies (nwex).

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They show that Ψ has the following properties. If $\pi = \Psi(\sigma)$, then

$$\begin{aligned}
\operatorname{des}\sigma &= \operatorname{wex}\pi - 1 \\
(31\text{-}2)\sigma &= A_{EE}(\pi) + A_{NN}(\pi) \\
(21\text{-}3)\sigma + (3\text{-}21)\sigma - \binom{\operatorname{des}\sigma}{2} &= A_{EN}(\pi) \\
(2\text{-}31)\sigma &= C_{EE}(\pi) + C_{NN}(\pi) \\
(1\text{-}32)\sigma + (32\text{-}1)\sigma - \binom{\operatorname{des}\sigma}{2} &= A_{NE}(\pi)
\end{aligned}$$

This implies:

- $A_{EN} + A_{NN}$ and $C_{EE} + C_{NN}$ are equidistributed,
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To see the latter note that the map $i \circ r \circ c$ (inverse of reversal of complement, or reflection across the antidiagonal of the permutation diagram) preserves wex, A_{EE}, A_{NN}, C_{EE}, C_{NN}, and exchanges A_{EN} and A_{NE}.

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Question: Describe the equivalent (under Φ) of irc on tableaux directly.

Catalan tableaux

- Recall that $|\mathfrak{S}_n(2\text{-}31)| = |\mathfrak{S}_n(2\text{-}3\text{-}1)| = C_n$, the nth Catalan number.

Catalan tableaux

- Recall that $|\mathfrak{S}_n(2\text{-}31)| = |\mathfrak{S}_n(2\text{-}3\text{-}1)| = C_n$, the nth Catalan number.
- Hence, (SW, 2005) C_n is the number of tableaux with no nontop 1s (i.e. with a single 1 per column).

Catalan tableaux

Theorem

If a tableau T has a single 1 per column, and $\pi = \Phi(T)$, then (the underlying sets of) the cycles of π form a noncrossing partition.

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- Suppose π contains two cycles c_i and c_j and elements $a > b > c > d$ such that a, c are in c_i and b, d are in c_j.

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- If $i < j$, then c_j contains c. If $i > j$, then c_i contains b. Neither is possible.
- Thus, the c_i's form a noncrossing partition. □

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- Hence, π avoids 3-2-1.

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- Then σ avoids 31-2 (i.e. 3-1-2).
- Note that the subsequences of weak excedance values of π and deficiency values of π are increasing.
- Hence, π avoids 3-2-1.

Theorem

$\mathfrak{S}_n(31\text{-}2) \xrightarrow{\Psi} \mathfrak{S}_n(3\text{-}2\text{-}1) \xleftarrow{\Phi}$ *monotone tableaux of semiperimeter n.*

Definition

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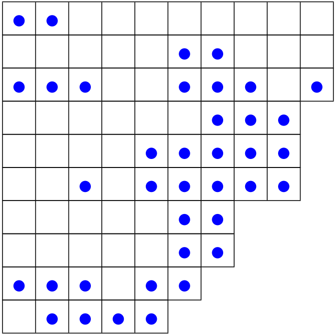
Example



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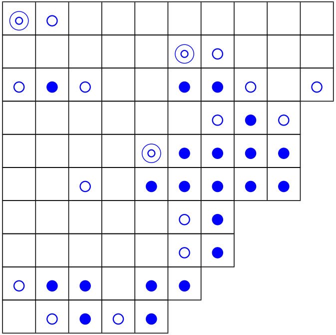

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Note that $n - ess(T) = dess(T) + zerorows(T)$.

Distribution

Conjecture (Steingrímsson, Williams, 2005)

$(n - ess, rows)$ has the same distribution on tableaux as $(cycles, wex)$ on permutations.

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Theorem (B., Eriksen, 2006)

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Each 0-row adds 1 to each of $dess + zerorows, rows, cycles, wex$, so we only need to consider tableaux with no 0-rows vs. derangements.

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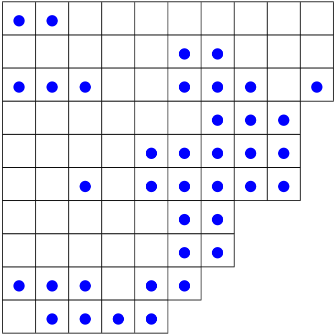

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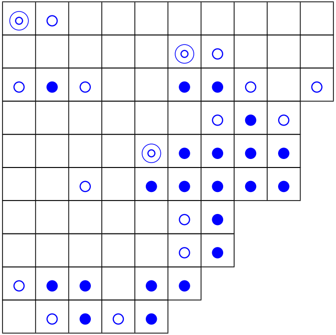

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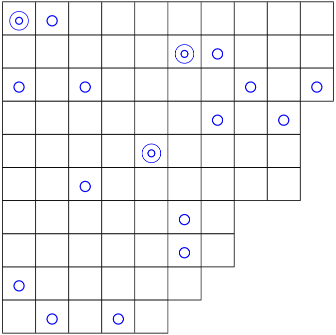

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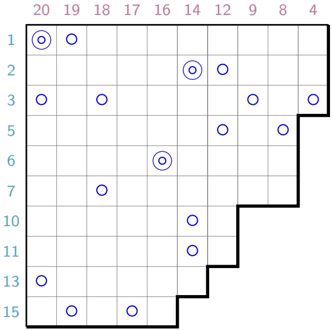

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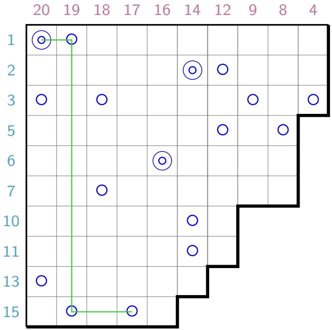

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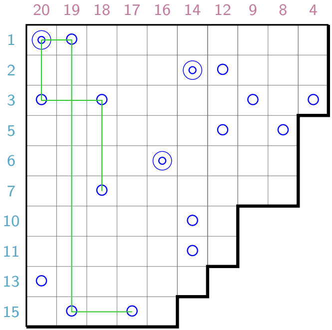

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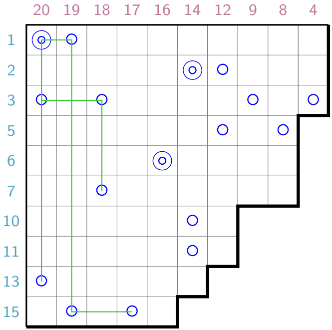

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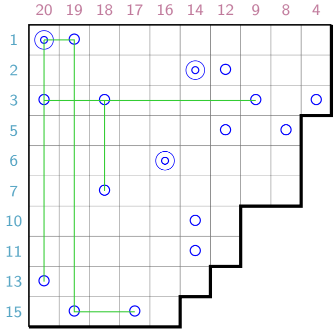

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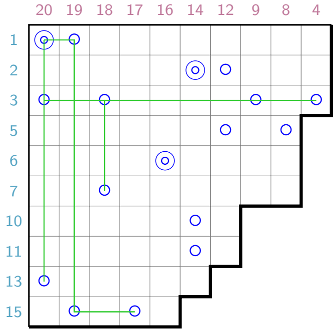

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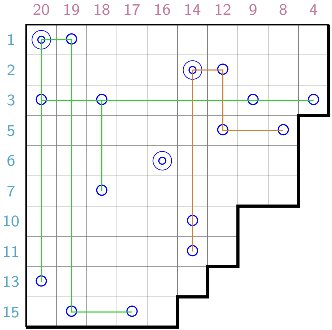

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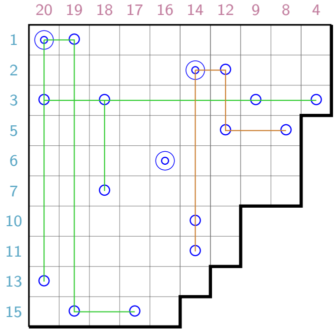

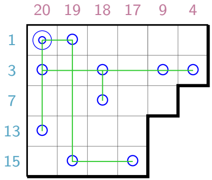

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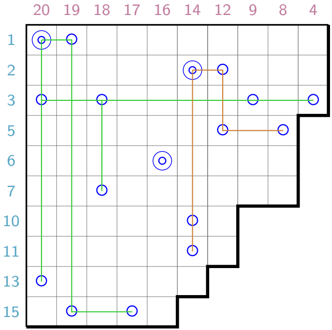

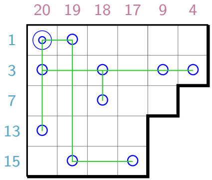

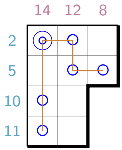

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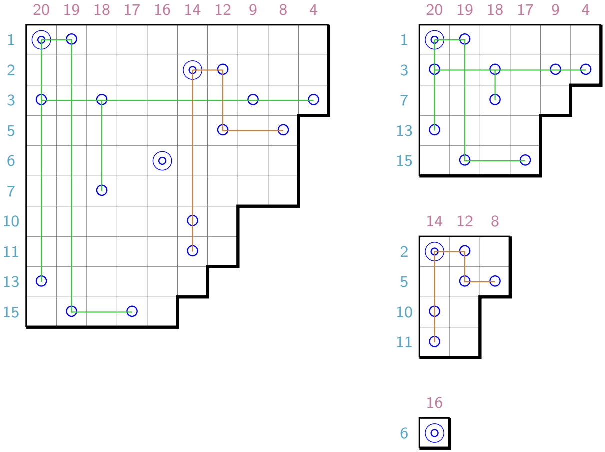

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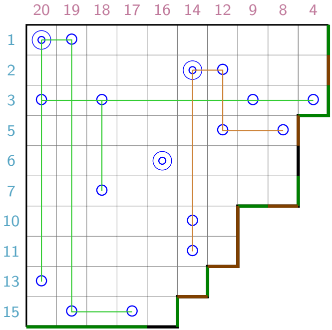

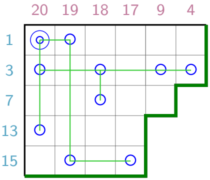

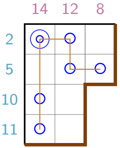

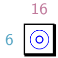

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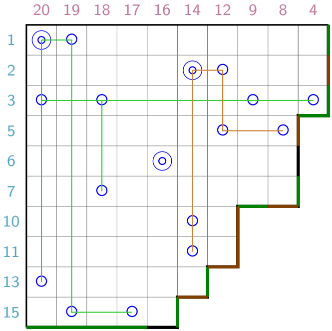

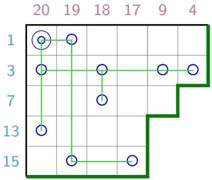

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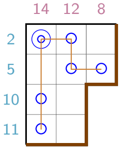

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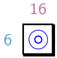
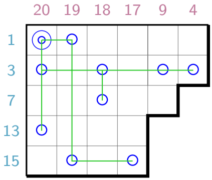


(6 16)

“Barely” labeled binary trees



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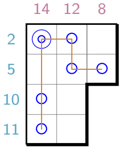

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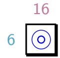

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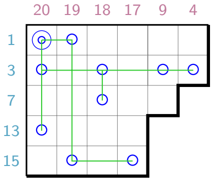

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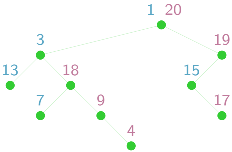

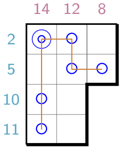

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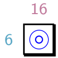

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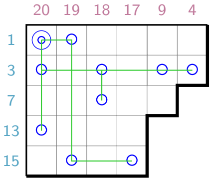

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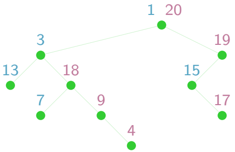

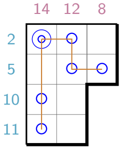

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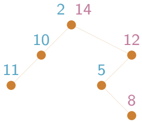

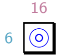

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- At the first return to the root, the next term is the largest label at the root. At the second return to the root (and when the root has no right child), the cycle is complete.

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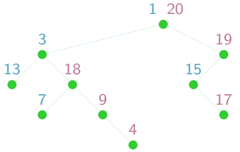

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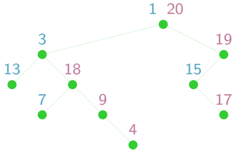

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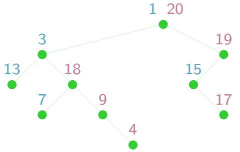

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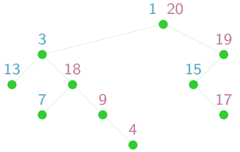

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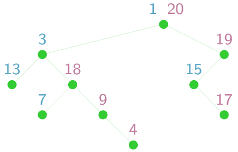

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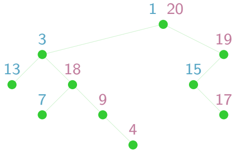

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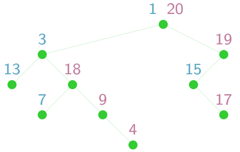

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- Etc.

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