Log-convexity of q-Catalan numbers

Lynne Butler
and undergraduate
Pat Flanigan at
Haverford College

The sequence of Catalan numbers

$$1, 1, 2, 5, 14, \ldots$$

is defined by $C_0 = 1$ and the recursion

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

Example: $C_3 = C_2 + C_1^2 + C_2 = 2 + 1^2 + 2.$

From the well-known formula

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

it is easily seen to be log-convex

$$(C_k)^2 \leq C_{k-1} C_{k+1}.$$

No formula is known for q-Catalan numbers.

Definition (Carlitz & Riordan, 1964):
Let $C_0(q) = 1$ and

$$C_{n+1}(q) = \sum_{k=0}^{n} q^{(k+1)(n-k)} C_k(q) C_{n-k}(q).$$

Example: $C_3(q) = q^2 C_2(q) + q^2 (C_1(q))^2 + C_2(q)$.

This recursion shows the sequence of these polynomials is increasing:

$C_0(q) = 1$

$C_1(q) = 1$

$C_2(q) = 1 + q$

$C_3(q) = 1 + q + 2q^2 + q^3$

$C_4(q) = 1 + q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + q^6$

To obtain a log-convexity result, we use the combinatorial interpretation:

$$C_k(q) = \sum_{\pi} q^{\mathsf{inv}\,\pi}$$

where the sum is over permutations with k 1s and k 2s such that every initial segment has no more 2s than 1s.

Example: $C_3(q) = 1 + q + 2q^2 + q^3$ because the permutations

$$111222, 112122, 121122, 112212, 121212$$

have inversion numbers 0,1,2,2,3 respectively.

Notice that $C_n(q)$ is monic of degree $\binom{n}{2}$, so

$$1 + \deg\,(C_k(q))^2 = \deg C_{k-1}(q)C_{k+1}(q).$$

Use this combinatorial interpretation to prove the log-convexity result:

Theorem (Butler & Flanigan, 2006): These q-Catalan numbers satisfy

$$q(C_k(q))^2 \leq C_{k-1}(q)C_{k+1}(q).$$

That is, $C_{k-1}(q)C_{k+1}(q) - q(C_k(q))^2$ **has non-negative coefficients.**

Example: The term qqq^3 of $qC_3(q)C_3(q)$

$$q(1 + q + 2q^2 + q^3)(1 + q + 2q^2 + q^3)$$

corresponds to the pair of permutations

$$\pi = 112122$$
$$\sigma = \quad 121212.$$

Our injection maps this pair to

$$\sigma_L \pi_R = \quad 1122$$
$$\pi_L \sigma_R = 11221212,$$

which corresponds to a term q^5 in $C_2(q)C_4(q)$.

Notice $1 + \operatorname{inv}\pi + \operatorname{inv}\sigma = \operatorname{inv}\sigma_L\pi_R + \operatorname{inv}\pi_L\sigma_R$.

More generally, for $1 \le r \le k$ and $\ell > k-r$,

$$q^{r(\ell-k+r)} C_k(q) C_\ell(q) \le C_{k-r}(q) C_{\ell+r}(q)$$

because there is an injection

$$\begin{aligned} \mathcal{P}_k \times \mathcal{P}_\ell &\to \mathcal{P}_{k-r} \times \mathcal{P}_{\ell+r} \\ (\pi, \sigma) &\mapsto (\sigma_L \pi_R, \pi_L \sigma_R) \end{aligned}$$

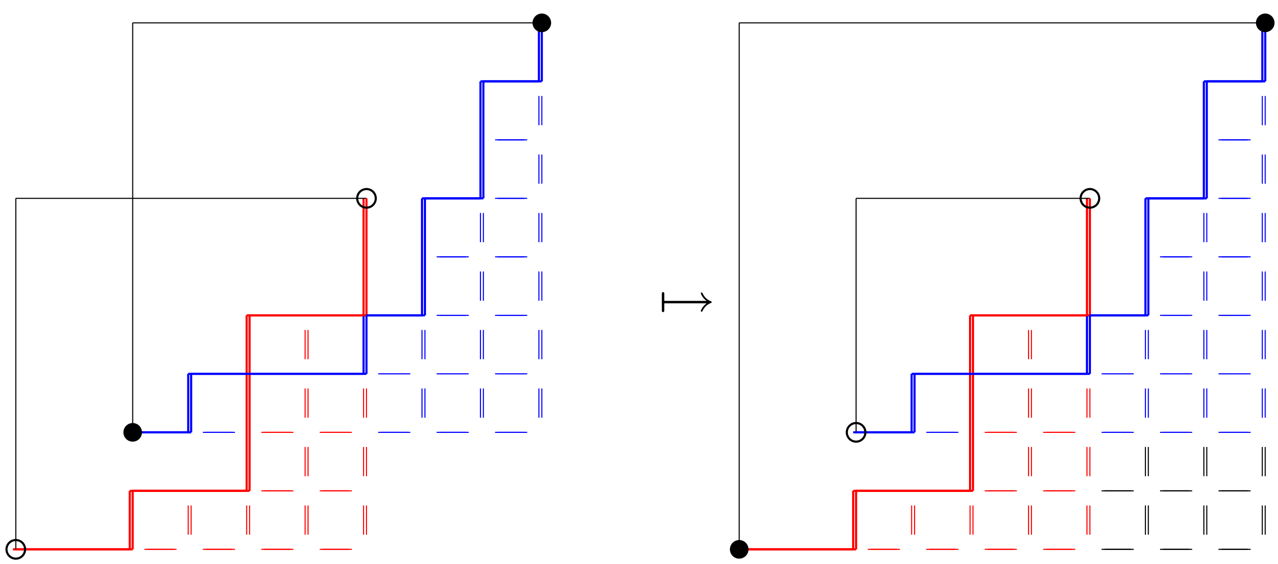
where $\mathcal{P}_n$ is the set of permutations with n 1s and n 2s such that every initial segment has no more 2s than 1s. Indent by $2r$ and calculate

$$r(\ell - k + r) + \mathrm{inv}\,\pi + \mathrm{inv}\,\sigma = \mathrm{inv}\,\sigma_L \pi_R + \mathrm{inv}\,\pi_L \sigma_R.$$

To see $q^{2(3)} C_6(q) C_7(q) \le C_4(q) C_9(q)$, visualize

112112221122 ↦ 12121122

    12111212212212   11211221121212212212



This undergraduate research with Flanigan was inspired by Butler's result that $\left[{n \atop 0}\right], \left[{n \atop 1}\right], \ldots, \left[{n \atop n}\right]$, the sequence of Gaussian polynomials, is log-concave. This result for the vector space ${F_q}^n$ may generalize to $Z/p^{\lambda_1}Z \times \cdots \times Z/p^{\lambda_\ell}Z$, which has $[\lambda, k]_p$ subgroups of order p^k.

Conjecture: $([\lambda, k]_p)^2 \geq [\lambda, k-1]_p\,[\lambda, k+1]_p$.

The fact that the sequence of coefficients in the Gaussian polynomial is unimodal, may also generalize.

Conjecture: The sequence of coefficients in the polynomial $[\lambda, k]_p$ is unimodal.

So, it is natural to ask about the q-Catalan numbers invented by Carlitz and Riordan:

Conjecture (Stanton): The sequence of coefficients in the polynomial $C_k(q)$ is unimodal.

$$C_5(q)=1+q+2q^2+3q^3+5q^4+5q^5+7q^6+7q^7+6q^8+4q^9+q^{10}$$

References:

[1] L. M. Butler, "A unimodality result in the enumeration of subgroups of a finite abelian group", *Proc. Amer. Math. Soc.* **101** (1987), 771–775.

[2] L. M. Butler, "The q-log-concavity of q-binomial coefficients", *J. Combin. Theory* **A54** (1990), 54–63.

[3] L. Carlitz and J. Riordan, "Two element lattice permutations and their q-generalization", *Duke J. Math.* **31** (1964), 371–388.

[4] D. Stanton, "Unimodality and Young's lattice", *J. Combin. Theory* **A54** (1990), 41-53.

[5] D. Zeilberger, "Kathy O'Hara's constructive proof of the unimodality of the Gaussian polynomials", *Amer. Math. Monthly* **96** (1989), 590–602.