Deodhar elements and embedded factor patterns

Brant C. Jones
brant@math.washington.edu

Joint with Sara C. Billey
University of Washington

June 13, 2006

The symmetric group is a Coxeter group

Fact

The symmetric group is generated by adjacent transpositions s_i *mapping*
$w(1)\cdots\mathbf{w(i)w(i+1)}\cdots w(n) \mapsto w(1)\cdots\mathbf{w(i+1)w(i)}\cdots w(n)$.

Example

$w = 3421$ can be obtained as
$1234 \xrightarrow{s_2} 1324 \xrightarrow{s_3} 1342 \xrightarrow{s_1} 3142 \xrightarrow{s_2} 3412 \xrightarrow{s_3} 3421$ and we write
$w = s_2 s_3 s_1 s_2 s_3$.

But there are also some **relations** like $s_1 s_2 s_1 = s_2 s_1 s_2$ and
$s_1 s_3 = s_3 s_1$.

The symmetric group is a Coxeter group

Fact

The symmetric group is generated by adjacent transpositions s_i *mapping*
$w(1)\cdots\mathbf{w(i)w(i+1)}\cdots w(n) \mapsto w(1)\cdots\mathbf{w(i+1)w(i)}\cdots w(n)$.

Example

$w = 3421$ can be obtained as
$1234 \xrightarrow{s_2} 1324 \xrightarrow{s_3} 1342 \xrightarrow{s_1} 3142 \xrightarrow{s_2} 3412 \xrightarrow{s_3} 3421$ and we write
$w = s_2s_3s_1s_2s_3$.

But there are also some **relations** like $s_1s_2s_1 = s_2s_1s_2$ and $s_1s_3 = s_3s_1$.

The symmetric group is a Coxeter group

Fact

The symmetric group is generated by adjacent transpositions s_i *mapping*
$w(1)\cdots \mathbf{w(i)w(i+1)}\cdots w(n) \mapsto w(1)\cdots \mathbf{w(i+1)w(i)}\cdots w(n)$.

Example

$w = 3421$ can be obtained as
$1234 \xrightarrow{s_2} 1324 \xrightarrow{s_3} 1342 \xrightarrow{s_1} 3142 \xrightarrow{s_2} 3412 \xrightarrow{s_3} 3421$ and we write
$w = s_2 s_3 s_1 s_2 s_3$.

But there are also some **relations** like $s_1 s_2 s_1 = s_2 s_1 s_2$ and
$s_1 s_3 = s_3 s_1$.

Definition

A *Coxeter system* (W, S) is a **group** W generated by $S = \{s_1, s_2, \cdots, s_p\}$ subject to the Coxeter **relations**:

$$s_i^2 = id$$

$$(s_i s_j)^{m_{ij}} = id$$

The data m_{ij} that determines the Coxeter group can be given as a labelled graph.

Example

The symmetric group has the Coxeter graph of type A:

$\bullet_{s_1} — \bullet_{s_2} — \bullet_{s_3} — \cdots \qquad \Longleftrightarrow$

$s_i s_{i\pm1} s_i = s_{i\pm1} s_i s_{i\pm1}$ and $s_i s_j = s_j s_i$ (for $|j - i| > 1$).

Definition

A *Coxeter system* (W, S) is a **group** W generated by $S = \{s_1, s_2, \cdots, s_p\}$ subject to the Coxeter **relations**:

$$s_i^2 = id$$

$$(s_i s_j)^{m_{ij}} = id$$

The data m_{ij} that determines the Coxeter group can be given as a labelled graph.

Example

The symmetric group has the Coxeter graph of type A:

$\bullet_{s_1} — \bullet_{s_2} — \bullet_{s_3} — \dots \qquad \Longleftrightarrow$

$s_i s_{i\pm1} s_i = s_{i\pm1} s_i s_{i\pm1}$ and $s_i s_j = s_j s_i$ (for $|j - i| > 1$).

Definition

Let (W, S) be a Coxeter system.

- An *expression* is any product of generators from S.
- The *length* $l(w)$ of an element $w \in W$, is the minimum length of any expression for the element w.
- A minimum length expression is called *reduced*. (e.g. $s_1 s_2 s_3 s_1 = s_1 s_2 s_1 s_3 = s_2 s_1 s_2 s_3$ is the permutation 3241.)

Theorem

(Tits, 1969) *Every reduced expression for w can be obtained by any other via a sequence of braid moves of the form* $\underbrace{stst\cdots}_{m(s,t)} \to \underbrace{tsts\cdots}_{m(s,t)}$.

Definition

Let (W, S) be a Coxeter system.

- An *expression* is any product of generators from S.
- The *length* $l(w)$ of an element $w \in W$, is the minimum length of any expression for the element w.
- A minimum length expression is called *reduced*. (e.g. $s_1 s_2 s_3 s_1 = s_1 s_2 s_1 s_3 = s_2 s_1 s_2 s_3$ is the permutation 3241.)

Theorem

(Tits, 1969) *Every reduced expression for w can be obtained by any other via a sequence of braid moves of the form* $\underbrace{stst\cdots}_{m(s,t)} \rightarrow \underbrace{tsts\cdots}_{m(s,t)}$.

Definition

Let (W, S) be a Coxeter system.

- An *expression* is any product of generators from S.
- The *length* $l(w)$ of an element $w \in W$, is the minimum length of any expression for the element w.
- A minimum length expression is called *reduced*. (e.g. $s_1 s_2 s_3 s_1 = s_1 s_2 s_1 s_3 = s_2 s_1 s_2 s_3$ is the permutation 3241.)

Theorem

(Tits, 1969) *Every reduced expression for w can be obtained by any other via a sequence of braid moves of the form* $\underbrace{stst\cdots}_{m(s,t)} \to \underbrace{tsts\cdots}_{m(s,t)}$.

Definition

Let (W, S) be a Coxeter system.

- An *expression* is any product of generators from S.
- The *length* $l(w)$ of an element $w \in W$, is the minimum length of any expression for the element w.
- A minimum length expression is called *reduced*. (e.g. $s_1 s_2 s_3 s_1 = s_1 s_2 s_1 s_3 = s_2 s_1 s_2 s_3$ is the permutation 3241.)

Theorem

(Tits, 1969) *Every reduced expression for w can be obtained by any other via a sequence of braid moves of the form*

$$\underbrace{stst\cdots}_{m(s,t)} \rightarrow \underbrace{tsts\cdots}_{m(s,t)}.$$

Definition

A *commutivity class* consists of all reduced words which can be obtained from each other by *commuting* braid relations of the form $st \to ts$ (i.e. where $m(s,t) = 2$).

Example

The commutivity classes of 3241 are $\{s_1 s_2 s_3 s_1, s_1 s_2 s_1 s_3\}$ and $\{s_2 s_1 s_2 s_3\}$.

Definition

The *support* of an element w is $supp(w) = \{$ generators s_i in any reduced expression for $w\}$. An element w is *connected* if $supp(w)$ is connected in the Coxeter graph.

Definition

A *commutivity class* consists of all reduced words which can be obtained from each other by *commuting* braid relations of the form $st \to ts$ (i.e. where $m(s,t) = 2$).

Example

The commutivity classes of 3241 are $\{s_1 s_2 s_3 s_1, s_1 s_2 s_1 s_3\}$ and $\{s_2 s_1 s_2 s_3\}$.

Definition

The *support* of an element w is $supp(w) = \{$ generators s_i in any reduced expression for $w\}$. An element w is *connected* if $supp(w)$ is connected in the Coxeter graph.

Definition

A *commutivity class* consists of all reduced words which can be obtained from each other by *commuting* braid relations of the form $st \rightarrow ts$ (i.e. where $m(s,t) = 2$).

Example

The commutivity classes of 3241 are $\{s_1 s_2 s_3 s_1, s_1 s_2 s_1 s_3\}$ and $\{s_2 s_1 s_2 s_3\}$.

Definition

The *support* of an element w is $supp(w) = \{$ generators s_i in any reduced expression for $w\}$. An element w is *connected* if $supp(w)$ is connected in the Coxeter graph.

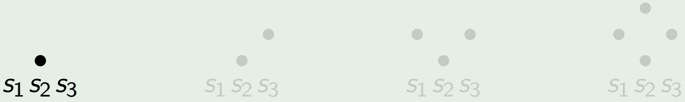
Given a reduced word w in a Coxeter group with Coxeter graph G, we can construct $Heap(w) \subset G \times \mathbb{N}$. The heap can also be regarded as a **labelled poset**, whose **total orderings** record **reduced expressions** for an element.

Fact

There is a distinct heap associated to each commutivity class of w.

Example

Fix a reduced word $w = s_2 s_3 s_1 s_2$ in type A_3:



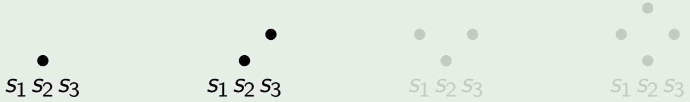
Given a reduced word w in a Coxeter group with Coxeter graph G, we can construct $Heap(w) \subset G \times \mathbb{N}$. The heap can also be regarded as a **labelled poset**, whose **total orderings** record **reduced expressions** for an element.

Fact

There is a distinct heap associated to each commutivity class of w.

Example

Fix a reduced word $w = s_2 s_3 s_1 s_2$ in type A_3:



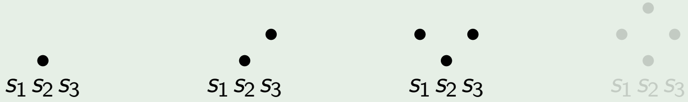
Given a reduced word w in a Coxeter group with Coxeter graph G, we can construct $Heap(w) \subset G \times \mathbb{N}$. The heap can also be regarded as a **labelled poset**, whose **total orderings** record **reduced expressions** for an element.

Fact

There is a distinct heap associated to each commutivity class of w.

Example

Fix a reduced word $w = s_2 s_3 s_1 s_2$ in type A_3:



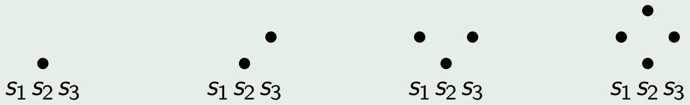
Given a reduced word w in a Coxeter group with Coxeter graph G, we can construct $Heap(w) \subset G \times \mathbb{N}$. The heap can also be regarded as a **labelled poset**, whose **total orderings** record **reduced expressions** for an element.

Fact

There is a distinct heap associated to each commutivity class of w.

Example

Fix a reduced word $w = s_2 s_3 s_1 s_2$ in type A_3:



Kazhdan-Lusztig polynomials

- Given a Coxeter group W, we can form the *Hecke algebra* $\mathcal{H}$ over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ that has basis $\{T_w : w \in W\}$, and relations:

$$T_s T_w = T_{sw} \text{ for } l(sw) > l(w)$$

$$(T_s)^2 = (q-1)T_s + qT_1$$

- There is another basis (defined to be invariant under a certain involution) called the *Kazhdan-Lusztig basis* consisting of $\{C'_w : w \in W\}$, and the *Kazhdan-Lusztig polynomials* arise as the "change of basis" matrix for these two bases:

$$C'_w = q^{-\frac{1}{2}l(w)} \sum_{x \leq w} P_{x,w}(q) T_x$$

Kazhdan-Lusztig polynomials

- Given a Coxeter group W, we can form the *Hecke algebra* $\mathcal{H}$ over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ that has basis $\{T_w : w \in W\}$, and relations:

$$T_s T_w = T_{sw} \text{ for } l(sw) > l(w)$$
$$(T_s)^2 = (q-1)T_s + qT_1$$

- There is another basis (defined to be invariant under a certain involution) called the *Kazhdan-Lusztig basis* consisting of $\{C'_w : w \in W\}$, and the *Kazhdan-Lusztig polynomials* arise as the "change of basis" matrix for these two bases:

$$C'_w = q^{-\frac{1}{2}l(w)} \sum_{x \leq w} P_{x,w}(q) T_x$$

Kazhdan-Lusztig polynomials

The Kazhdan-Lusztig basis elements multiply as:

$$C'_s C'_w = C'_{sw} + \sum_{sz<z<w} \bar{\mu}(z,w) q^{\frac{1}{2}(l(w)-l(z))} C'_z$$

Compare this with the simpler formula:

$$T_s T_w = T_{sw}$$

so that if $a = a_1 a_2 \cdots a_p$ is a reduced expression for w,

$$T_w = T_{a_1} T_{a_2} \cdots T_{a_p}$$

Definition

We set $P_{w,w}(q) = 1$ and $P_{x,w}(q) = 0$ if $x \not\leq w$ in Bruhat order. (i.e. $x \leq w$ in Bruhat order if there is a reduced expression for w that contains a reduced expression for x as a (not necessarily consecutive) subword.)
Then, for $\mathbf{ws} < \mathbf{w}$:

$$P_{x,w} = P_{xs,w} \textbf{ if xs} > \mathbf{x}$$

$$P_{x,w} = P_{xs,ws} + qP_{x,ws} - \sum_{z:zs<z} q^{\frac{l(w)-l(z)}{2}} \bar{\mu}(z, ws) P_{x,z} \textbf{ if xs} < \mathbf{x}$$

where $\bar{\mu}(z, ws)$ is the coefficient of (highest possible degree) $q^{\frac{l(ws)-l(z)-1}{2}}$ in $P_{z,ws}(q)$.

Computing Kazhdan-Lusztig polynomials

Deodhar's (1990) combinatorial framework:

$$P_{x,w}(q) = \sum_{\substack{\sigma \in E_w \\ \pi(\sigma)=x}} q^{d(\sigma)}$$

where E_w is a set of subexpressions σ of w called *masks*, and d is a combinatorial statistic on masks called the *defect* of the mask.

Warning: The set E_w of masks which produce the Kazhdan-Lusztig polynomial is defined *recursively*, in general.

Example

Deodhar's combinatorial framework (by example):
$P_{x,w} = P_{s_2 s_3, s_2 s_3 s_4 s_1 s_2 s_3} = 1 + 2q$ since we have masks:

s_2	s_3	s_4	s_1	s_2	s_3	defect zeros	non-defect zeros
1	1	0	0	0	0	1	3
1	0	0	0	0	1	1	3
0	0	0	0	1	1	0	4

Definition

An element w of a Weyl group W is *Deodhar* if for all masks σ on any reduced expression of w, we have:

$$\# \text{ defect zeros of } \sigma < \# \text{ non-defect zeros of } \sigma$$

Theorem

(Deodhar, 1990) *Let W be any finite Weyl group and $a = a_1 a_2 \cdots a_p$ be a reduced expression for some $w \in W$. If w is Deodhar, then:*

- *The Kazhdan-Lusztig polynomial $P_{x,w}$ for all $x \in W$ is*

$$P_{x,w} = \sum_{\text{all masks } \sigma \text{ such that } a^{\sigma}=x} q^{\#\text{ defects of } \sigma}$$

- *The Kazhdan-Lusztig basis element $C'_w = q^{-\frac{1}{2}l(w)} \sum_{\text{all masks } \sigma \text{ of } w} q^{\#\text{ defects of } \sigma} T_{a^{\sigma}}$.*
- *The Kazhdan-Lusztig basis element C'_w satisfies:*

$$C'_w = C'_{a_1} C'_{a_2} \cdots C'_{a_p}$$

Theorem

(Deodhar, 1990) *Let W be any finite Weyl group and $a = a_1 a_2 \cdots a_p$ be a reduced expression for some $w \in W$. If w is Deodhar, then:*

- *The Kazhdan-Lusztig polynomial $P_{x,w}$ for all $x \in W$ is*

$$P_{x,w} = \sum_{\text{all masks } \sigma \text{ such that } a^{\sigma}=x} q^{\#\text{ defects of } \sigma}$$

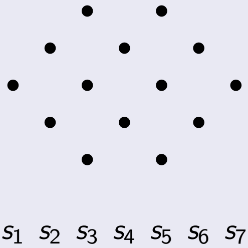
- *The Kazhdan-Lusztig basis element $C'_w = q^{-\frac{1}{2}l(w)} \sum_{\text{all masks } \sigma \text{ of } w} q^{\#\text{ defects of } \sigma} T_{a^{\sigma}}$.*
- *The Kazhdan-Lusztig basis element C'_w satisfies:*

$$C'_w = C'_{a_1} C'_{a_2} \cdots C'_{a_p}$$

Theorem

(Billey-Warrington, 2001) *In type A, an element is Deodhar if and only if it avoids 321 and it heap-avoids the hexagon pattern:*

$w = s_5 s_6 s_7 s_3 s_4 s_5 s_6 s_2 s_3 s_4 s_5 s_1 s_2 s_3$



This result extends to all linear Coxeter types.

Enumerative results

- Stankova and West (2004) enumerated the 321-hexagon avoiding permutations. They found that there is a 7-term constant coefficient linear recurrance.
- Monsour and Stankova (2002/3) enumerated the 321-$(2k)$-gon avoiding permutations for all k.
- Vatter (2005) found an enumeration scheme for the 321-hexagon avoiding permutations.

Definition

Let w and p be Coxeter elements, with p connected.
A *Coxeter embedding of p* is an injective map
$f : supp(p) \rightarrow supp(w)$ that respects the Coxeter relations.
We say w *contains p as an embedded factor* if there exists a reduced expression for w that contains a Coxeter embedding of a reduced expression for p as a consecutive subword.

Example

In type A, $w = s_2 s_4 s_3 s_2$ contains the pattern $p = s_2 s_1 s_2$ since $w = s_4 s_2 s_3 s_2 = s_4 \mathbf{s_3 s_2 s_3}$, and $f : \{s_1, s_2\} \mapsto \{s_2, s_3\}$ is a Coxeter embedding.

Theorem

(Billey-Jones, 2006) *In any Coxeter group, if p is not Deodhar and w contains p as an embedded factor, then w is not Deodhar either.*

Corollary

Deodhar elements in any *Coxeter group must be short-braid avoiding. Hence, they are fully-commutative and so they have a unique heap.*

s_1	s_2	s_1
1	0	0

Theorem

(Billey-Jones, 2006) *In any Coxeter group, if p is not Deodhar and w contains p as an embedded factor, then w is not Deodhar either.*

Corollary

Deodhar elements in any *Coxeter group must be short-braid avoiding. Hence, they are fully-commutative and so they have a unique heap.*

s_1 s_2 s_1

1 0 0

Example

Suppose we want to avoid $p = s_1 s_2 s_1 s_3 = 3241$ from S_4 as an **embedded factor**. Then, elements of S_4 like:

- $(s_1 s_2 s_1 s_3) s_2 = 3421$,
- $s_3 (s_1 s_2 s_1 s_3) = 4231$, and
- $s_3 (s_1 s_2 s_1 s_3) s_2 = 4321$

all contain p as factors, so they must be included in any list of elements to avoid as **permutation patterns**.

Definition

If avoiding p as an embedded factor is *equivalent* to avoiding the set of up-ideals in 2-sided weak order generated by Coxeter embeddings of p, then we say p is *ideal*.

Example

Suppose we want to avoid $p = s_1 s_2 s_1 s_3 = 3241$ from S_4 as an **embedded factor**. Then, elements of S_4 like:

- $(s_1 s_2 s_1 s_3) s_2 = 3421$,
- $s_3 (s_1 s_2 s_1 s_3) = 4231$, and
- $s_3 (s_1 s_2 s_1 s_3) s_2 = 4321$

all contain p as factors, so they must be included in any list of elements to avoid as **permutation patterns**.

Definition

If avoiding p as an embedded factor is *equivalent* to avoiding the set of up-ideals in 2-sided weak order generated by Coxeter embeddings of p, then we say p is *ideal*.

Theorem

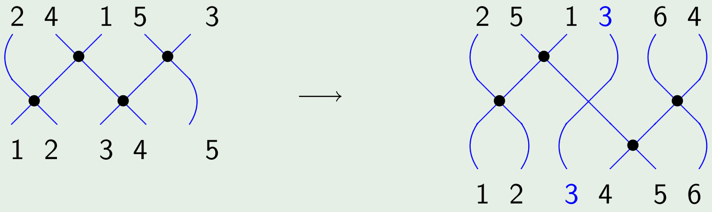
(Tenner, 2005) *In type A, if p is connected and 2143-avoiding, then p is an ideal pattern.*

Example

The pattern $s_1 s_2 s_1 = [321]$ is ideal. Hence, w is fully commutative
$\iff$ w avoids $s_1 s_2 s_1$ as an **embedded factor**
$\iff$ w avoids $[321]$ as a **permutation pattern**.
Originally proved by Billey-Jockusch-Stanley (1993).

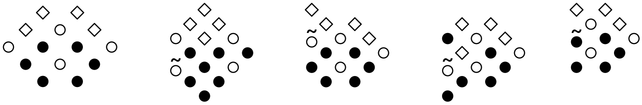
Example

Consider that $w = \underline{251}3\underline{64}$ contains $p = 24153$ as a permutation pattern. However, none of the reduced expressions for $w = [s_4 s_5 s_3 s_1 s_2]$ contain $p = [s_3 s_4 s_1 s_2]$ as reduced expression patterns.

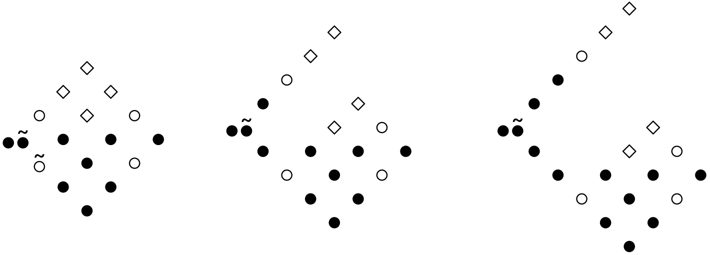


Theorem

(Billey-Jones, 2006) *The complete list of minimal non-Deodhar short-braid avoiding embedded factor patterns for type D is given by* HEX, HEX_2, HEX_{3a}, HEX_{3b}, HEX_5 *along with* $FLHEX_k$ *for all* $k \geq 0$.



HEX $\quad$ HEX_2 $\quad$ HEX_{3a} $\quad$ HEX_{3b} $\quad$ HEX_5



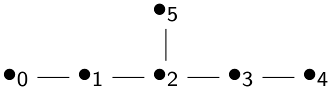
$FLHEX_0$ $FLHEX_1$ $FLHEX_2$ $\cdots$

All of these contain $[\bar{1}, 6, 7, 8, \bar{5}, 2, 3, 4]$ as 1-line patterns.

Heaps are shown with masks which demonstrate the non-Deodhar condition: $\diamond$ =defect, $\circ$ =mask value 0, $\bullet$ =mask value 1

Results

The other semisimple types are finite so computable by exhaustion. . . In types E_6 there are 4 patterns (besides 321-avoiding):



$s_0 s_1 s_2 s_5 s_3 s_4 s_2 s_3 s_1 s_2 s_5 s_0 s_1$

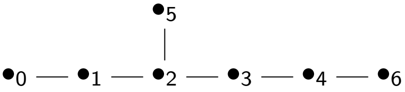
$s_5 s_1 s_2 s_3 s_0 s_1 s_2 s_5 s_4 s_3 s_2 s_1 s_0$

$s_1 s_2 s_5 s_3 s_4 s_2 s_3 s_1 s_2 s_5 s_0 s_1 s_2$

$s_2 s_5 s_1 s_2 s_3 s_0 s_1 s_2 s_5 s_4 s_3 s_2 s_1$

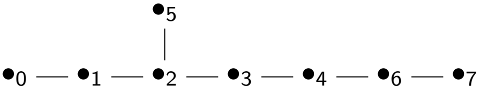
Results

In E_7 there are 5 new patterns:



$$s_0 s_1 s_2 s_3 s_4 s_6 s_5 s_2 s_3 s_4 s_1 s_2 s_3 s_0 s_1$$
$$s_3 s_4 s_6 s_1 s_2 s_3 s_0 s_1 s_2 s_5 s_4 s_3 s_2 s_1 s_0$$
$$s_1 s_2 s_3 s_4 s_6 s_5 s_2 s_3 s_4 s_1 s_2 s_3 s_0 s_1 s_2$$
$$s_2 s_3 s_4 s_6 s_1 s_2 s_3 s_0 s_1 s_2 s_5 s_4 s_3 s_2 s_1$$
$$s_5 s_2 s_3 s_4 s_6 s_1 s_2 s_5 s_3 s_4 s_2 s_3 s_0 s_1 s_2 s_5$$

In E_8 there are no new patterns:



Enumerating Deodhar elements in Weyl Groups

Type/Rank	2	3	4	5	6	7	8
A/B/G/F	5	14	42	132	429	1426/ 1430	4806/ 4862
D			48	167	575/ 593	1976/ 2144	6791/ 7864
E					642/ 662	2341/ 2670	8305/ 10846

Number of Deodhar elements shown as a fraction of the fully-commutative elements enumerated by Stembridge (1998).

Related topics / open directions

- Consider root subsystem patterns of Billey, Postnikov, Braden.
- Enumeration of emebedded factor pattern classes, especially for the type D Deodhar elements.
- Mansour-West (2004) have enumerated the 1-line pattern classes in type B_n with basis from B_2.
- Game of K. Eriksson used in computation.
- Characterize the minimal admissible sets of Deodhar.