Interval Avoidance in the Symmetric Group

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(joint work with Alexander Woo, UC Davis)

Permutation Embeddings

Definition (Embedding of pattern π into σ)

*Given $\pi = \pi_1\pi_2\cdots\pi_m \in \mathfrak{S}_m$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$, an **embedding** is a choice of indices $i_1 < i_2 < \cdots < i_m$ such that $\sigma_{i_j} < \sigma_{i_k}$ if and only if $\pi_j < \pi_k$ for each $j, k = 1, 2, \ldots, m$:*

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Notation (Avoidance Set for a pattern π)

$S_n(\pi) = \{\sigma \in \mathfrak{S}_n \mid \sigma$ *does not contain an embedding of* $\pi\}$

The (Strong) Bruhat Order

Definition (Inversion in $\sigma \in \mathfrak{S}_n$)

An inversion is an embedding of the pattern $21 \in \mathfrak{S}_2$ into σ.

Definition (Length Function on $\mathfrak{S}_n$)

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Bruhat Covering Relation

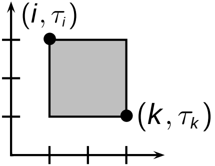
Definition (Bruhat covering relation on $\mathfrak{S}_n$)

We say $\sigma \prec \tau$ in Bruhat order if

- $\sigma = \tau t$ *for some transposition* t
- $\ell(\sigma) = \ell(\tau) - 1$

Equivalently: use transposition t to "undo" an embedding of 21

- at positions $i < k$ in τ such that
- $\nexists$ index j for which $i < j < k$ and $\tau_i > \tau_j > \tau_k$:



Symmetry Properties of Bruhat Order

Lemma (Bruhat order symmetries for $\sigma, \tau \in \mathfrak{S}_n$)

- *(Inverses)* $\sigma < \tau \implies \sigma^{-1} < \tau^{-1}$
- *(Reverse)* $\sigma < \tau \implies \tau^r < \sigma^r$
- *(Complement)* $\sigma < \tau \implies \tau^c < \sigma^c$
- *(Reverse Complement)* $\sigma < \tau \implies \sigma^{rc} < \tau^{rc}$

Examples: Starting with $1324 < 2341$,

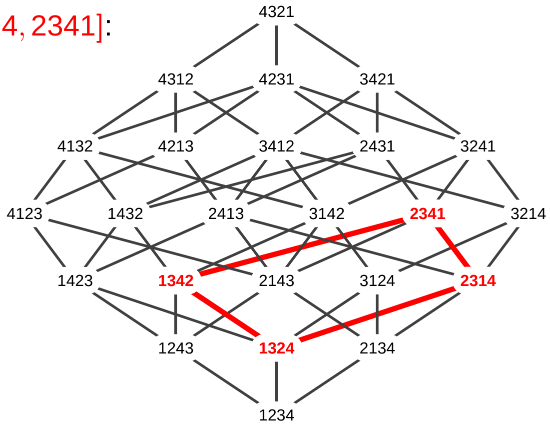
- $1324^{-1} = 1324 < 4123 = 2341^{-1}$.
- $2341^r = 1432 < 4231 = 1324^r$.
- $2341^c = 3214 < 4231 = 1324^c$.
- $1324^{rc} = 1324 < 4123 = 2341^{rc}$.

Intervals in Bruhat Order

Definition (Intervals in Bruhat order)

Given $\sigma, \tau \in \mathfrak{S}_n$, $[\sigma, \tau] = \{\omega \in \mathfrak{S}_n \mid \sigma \leq \omega \leq \tau\}$.

Example. $[1324, 2341]$:



Embeddings Intervals into Larger Intervals

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Given $\pi \leq \rho \in \mathfrak{S}_m$ and $\sigma \leq \tau \in \mathfrak{S}_n$ with $m \leq n$, we say that $[\pi, \rho]$ embeds into $[\sigma, \tau]$ if

$$\left.\begin{array}{l} \bullet\ \pi \text{ embeds into } \sigma \\ \bullet\ \rho \text{ embeds into } \tau \end{array}\right\} \text{using same embedding } (i_1, i_2, \ldots, i_m)$$

- *the intervals $[\pi, \rho]$ and $[\sigma, \tau]$ are order-isomorphic.*

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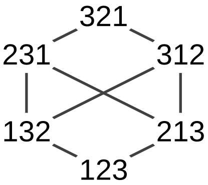

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- *π embeds into σ*
- *ρ embeds into τ*

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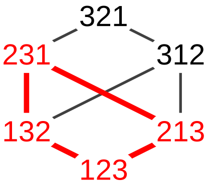

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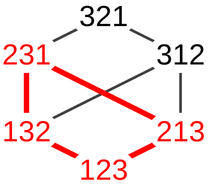
Given $\pi \leq \rho \in \mathfrak{S}_m$ and $\sigma \leq \tau \in \mathfrak{S}_n$ with $m \leq n$, we say that $[\pi, \rho]$ embeds into $[\sigma, \tau]$ if

$$\left.\begin{array}{l} \bullet\ \pi \text{ embeds into } \sigma \\ \bullet\ \rho \text{ embeds into } \tau \end{array}\right\} \text{using same embedding } (i_1, i_2, \ldots, i_m)$$

- *the intervals $[\pi, \rho]$ and $[\sigma, \tau]$ are order-isomorphic.*

Example. $[123, 231]$ embeds into $[1324, 2341]$:



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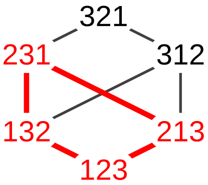

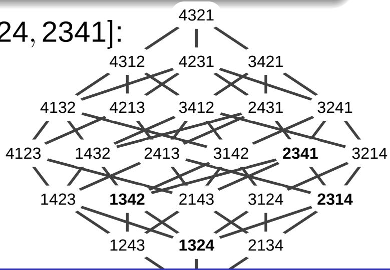

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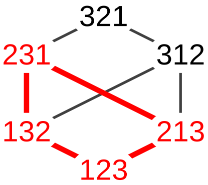

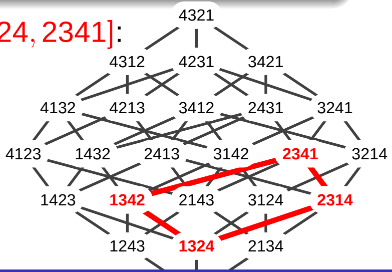

An Equivalent Definition of Interval Embeddings

Lemma (Interval Embedding Characterization)

Given $\pi \leq \rho \in \mathfrak{S}_m$ *and* $\sigma \leq \tau \in \mathfrak{S}_n$ *with* $m \leq n$*, the interval* $[\pi, \rho]$ *embeds into* $[\sigma, \tau]$ *iff*

- $\sigma_i = \tau_i$ *for* $i \notin \{i_1, i_2, \ldots, i_m\}$ *(a common embedding)*
- $\ell(\tau) - \ell(\sigma) = \ell(\rho) - \ell(\pi)$

Corollary

Given any three of the permutations π*,* ρ*,* σ*, and* τ*, the fourth is uniquely determine.*

Definition (Avoidance Set for an Interval)

$S_n([\pi, \rho]) = \{\tau \in \mathfrak{S}_n \mid \forall \sigma \in \mathfrak{S}_n,\ [\pi, \rho]$ *doesn't embed into* $[\sigma, \tau]\}$.

Examples of Interval Embeddings & Avoidance

Examples:

- If $\pi = \rho$, then $S_n([\pi, \rho]) = S_n(\rho)$ since the intervals $[\pi, \rho] = \{\rho\}$ and $[\sigma, \tau] = \{\tau\}$ are trivially order-isomorphic.
- 43512 "contains" $[1324, 3412]$ because the interval $[1324, 3412]$ embeds into $[4\mathbf{1325}, 4\mathbf{3512}]$:

  $$\ell(43512) - \ell(41325) = 7 - 4 = 4 - 1 = \ell(3412) - \ell(1324)$$

- $426153 \in S_n([1324, 3412])$ because the interval $[1324, 3412]$ cannot embed into $[\mathbf{12}4\mathbf{3}56, \mathbf{426}15\mathbf{3}]$:

  $$\ell(426153) - \ell(124356) = 8 - 1 > 4 - 1 = \ell(3412) - \ell(1324)$$

- "Universal" in characterizing singularities of Schubert varieties (A. Woo and A. Yong).

A Geometric Form for Interval Pattern Containment

Algorithm (Forbidden Region for $\pi \leq \rho$)

- *Graph π as circles, ρ as dots.*
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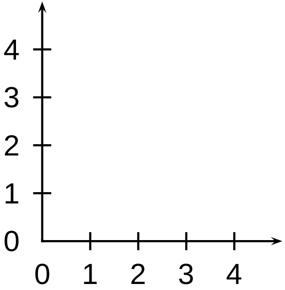

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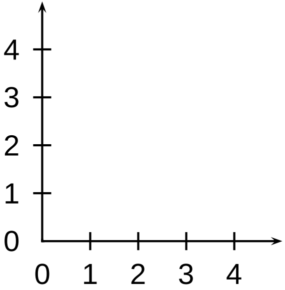

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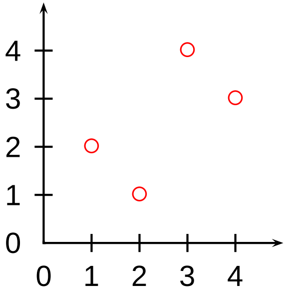

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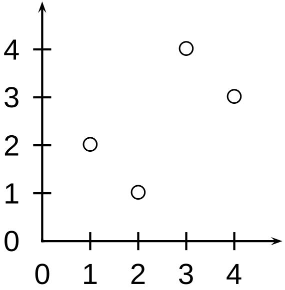

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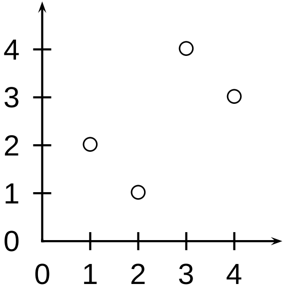

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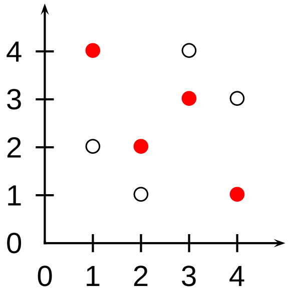

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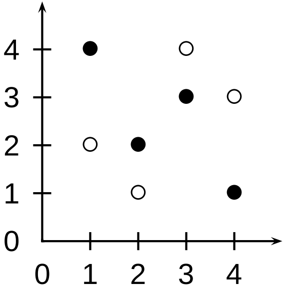

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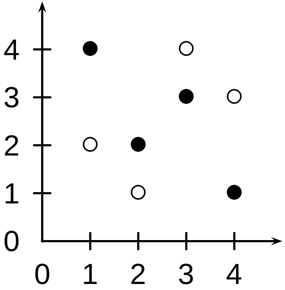

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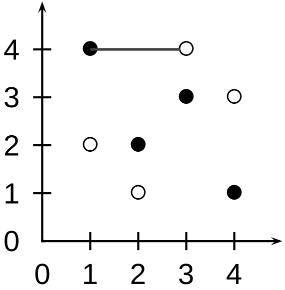

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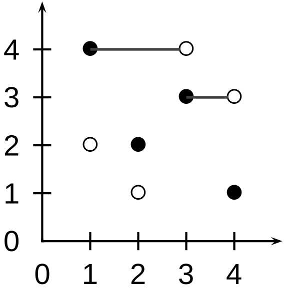

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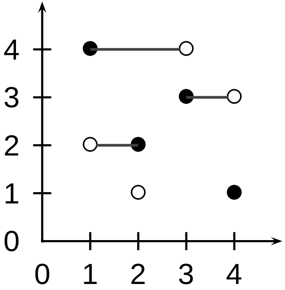

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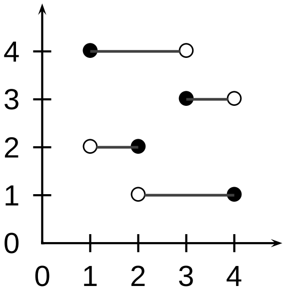

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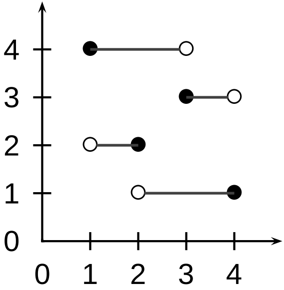

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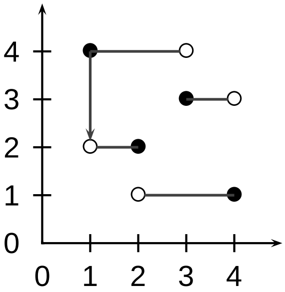

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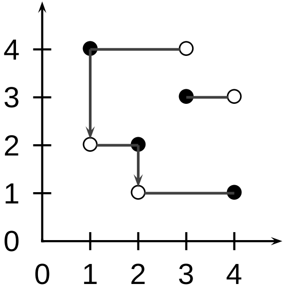

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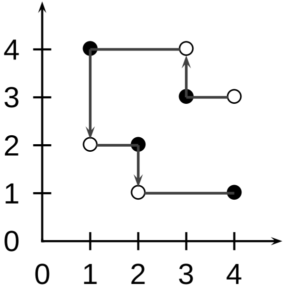

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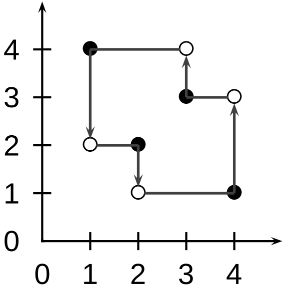

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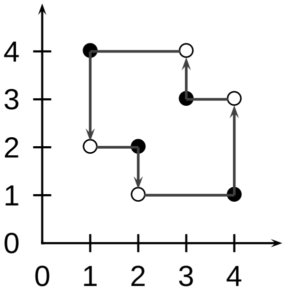

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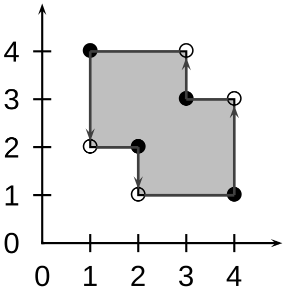

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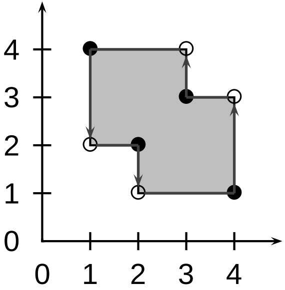

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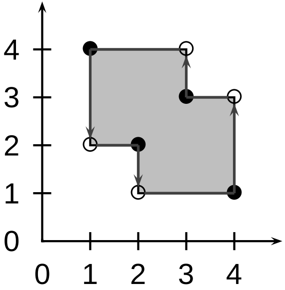


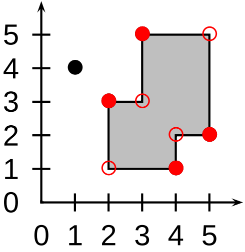
Lemma

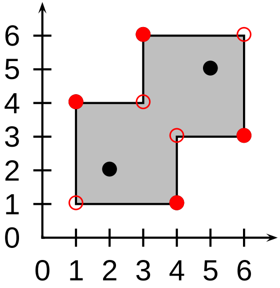
Then a permutation $\tau \in \mathfrak{S}_n$ "contains" $[\pi, \rho]$ iff the forbidden region constructed above contains no "non-embedding" points.

Examples of Forbidden Regions

Examples:

- 43512 “contains” $[1324, 3412]$ because the Forbidden Region contains no “non-embedding” points.
- $426153 \in S_n([1324, 3412])$ because the Forbidden Region contains “non-embedding” points.





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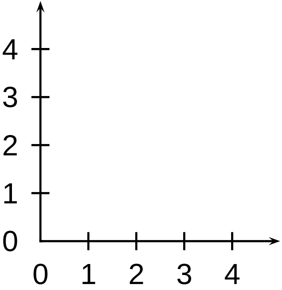

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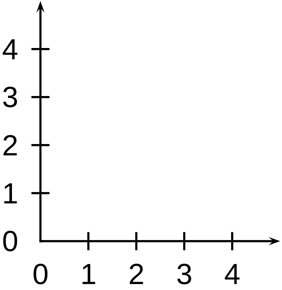

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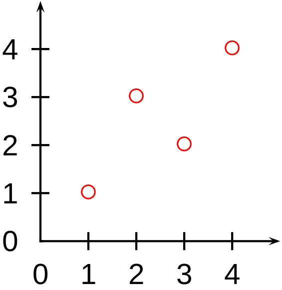

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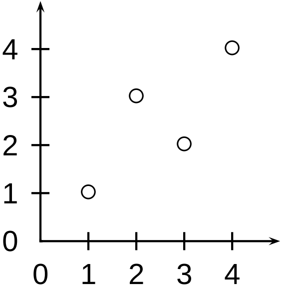

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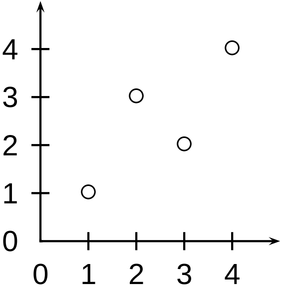

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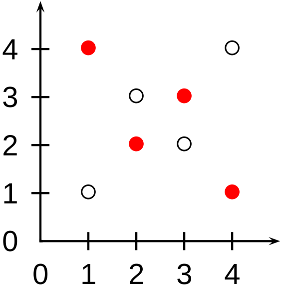

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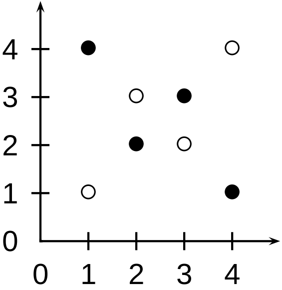

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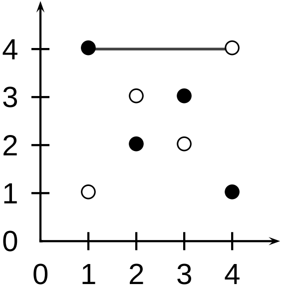

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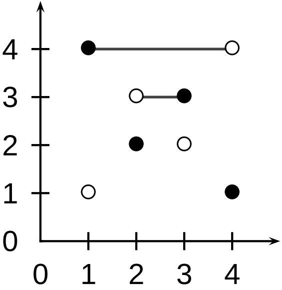

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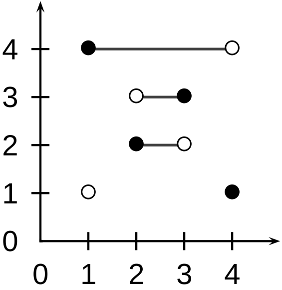

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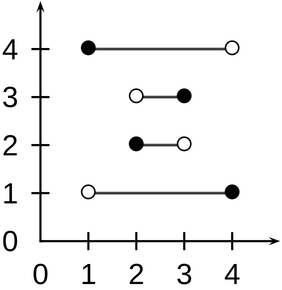

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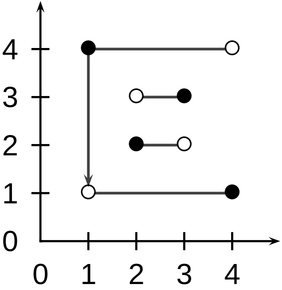

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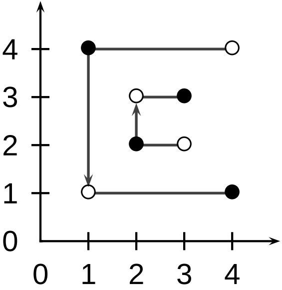

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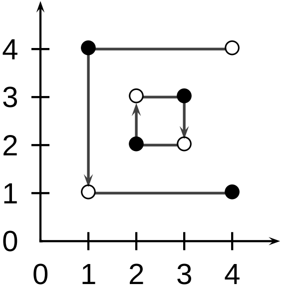

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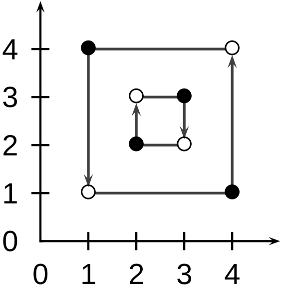

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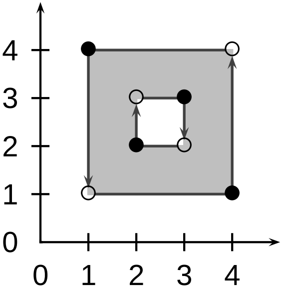

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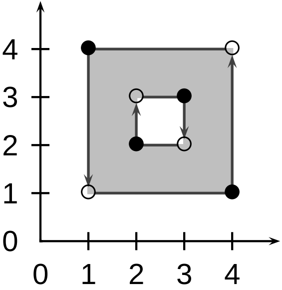

For $[3412, 4321]$:

Examples of Strange Forbidden Regions

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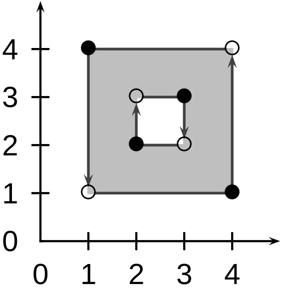

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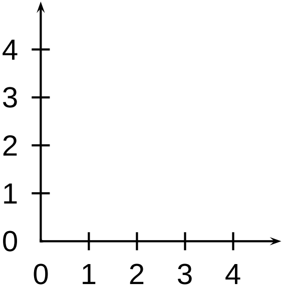

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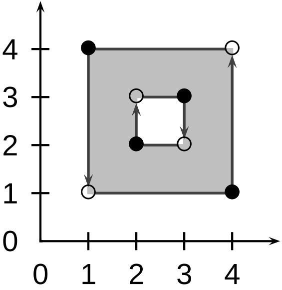

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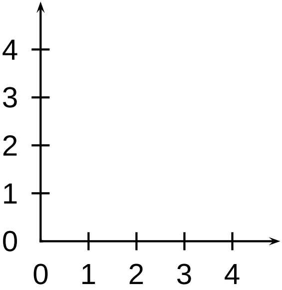

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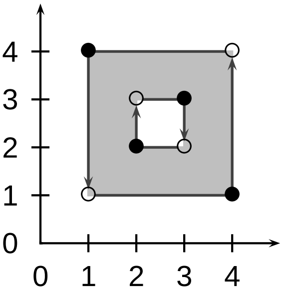

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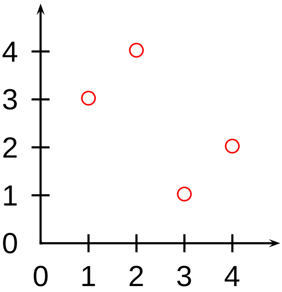

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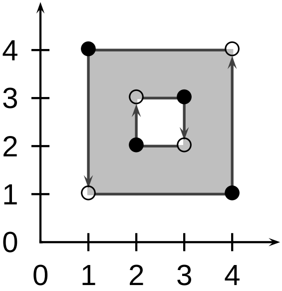

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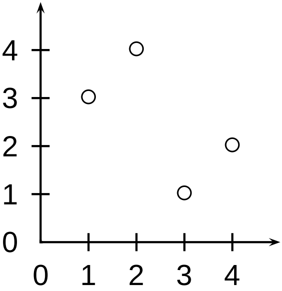

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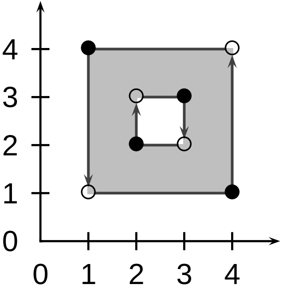

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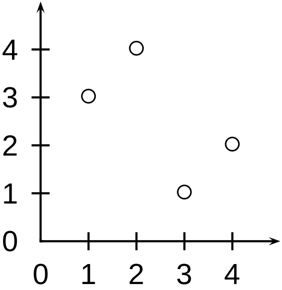

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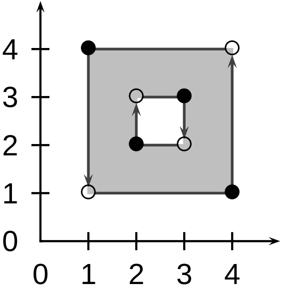

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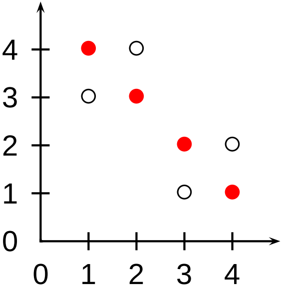

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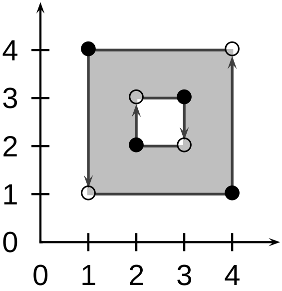

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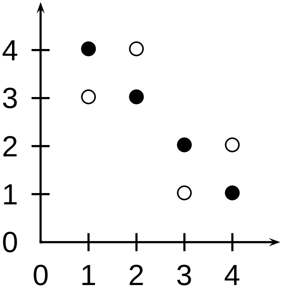

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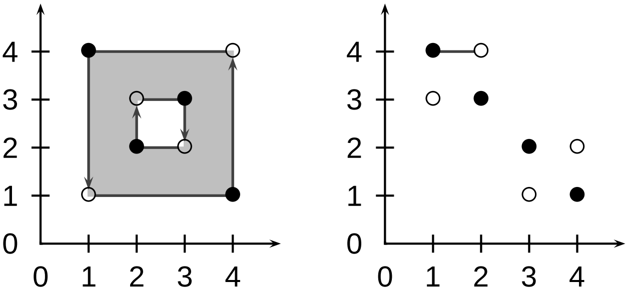

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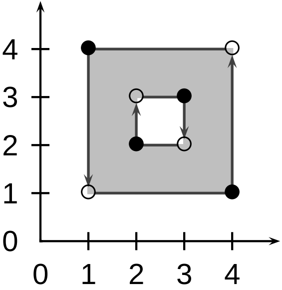

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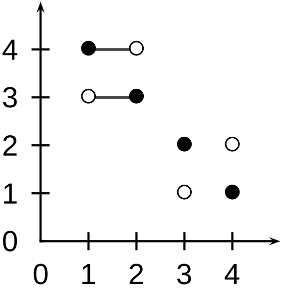

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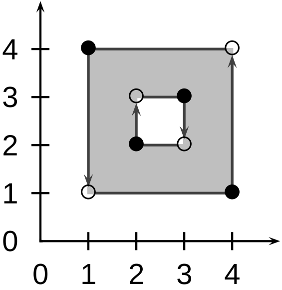

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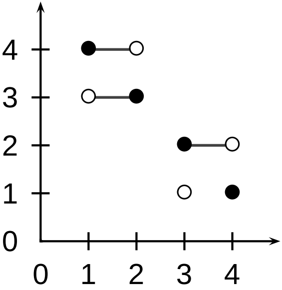

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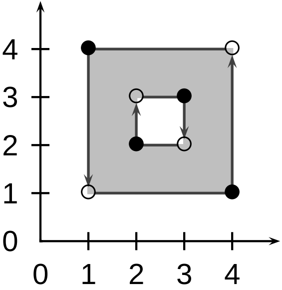

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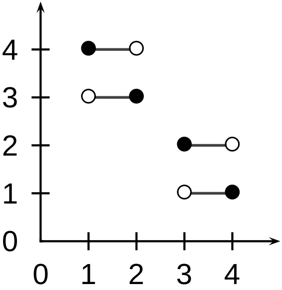

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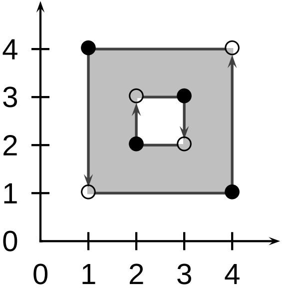

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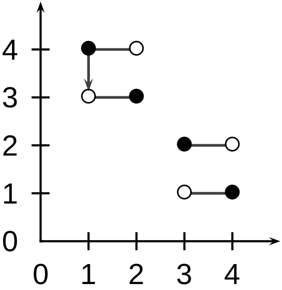

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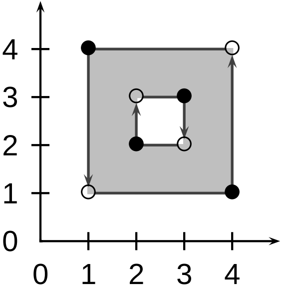

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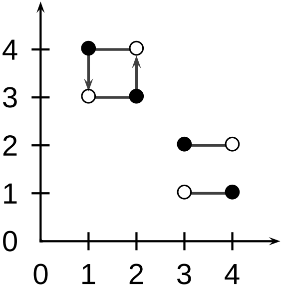

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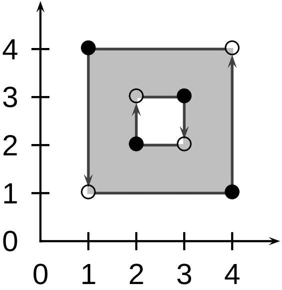

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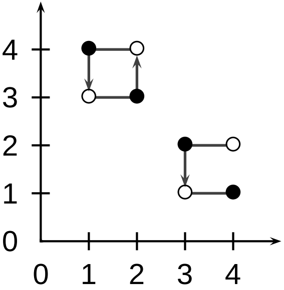

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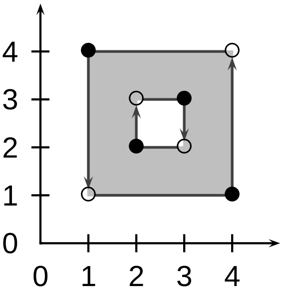

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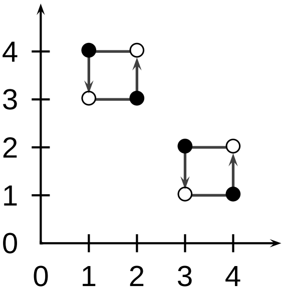

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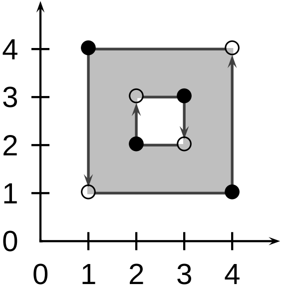

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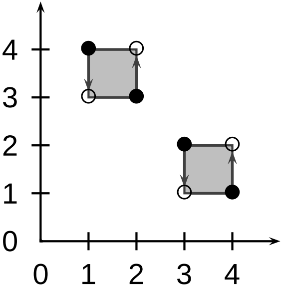

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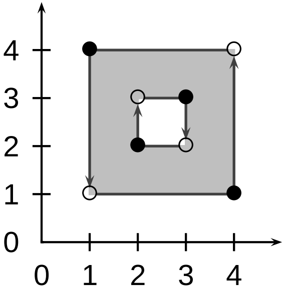

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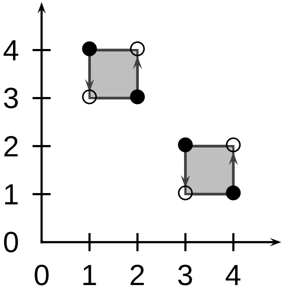

For $[123, 321]$:

Examples of Strange Forbidden Regions

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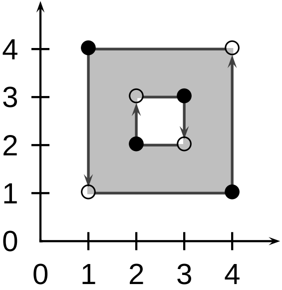

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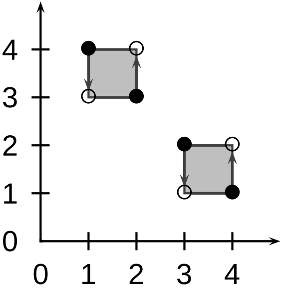

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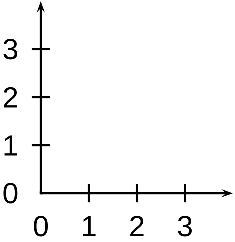

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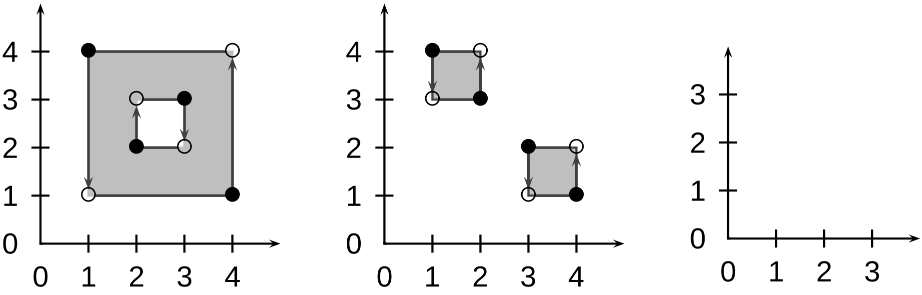

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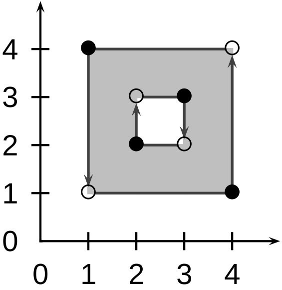

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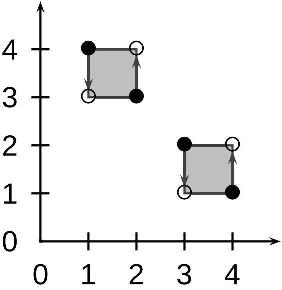

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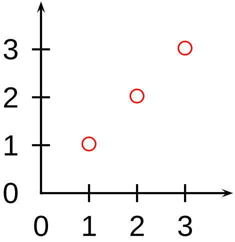

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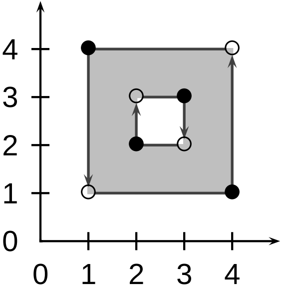

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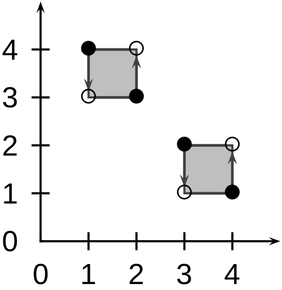

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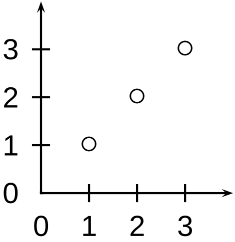

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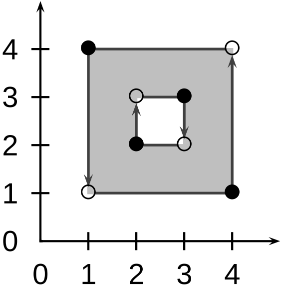

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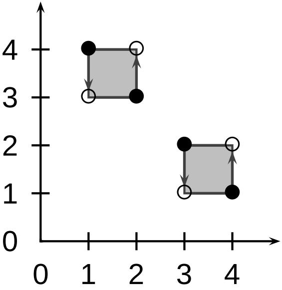

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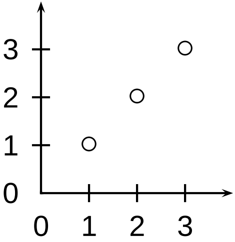

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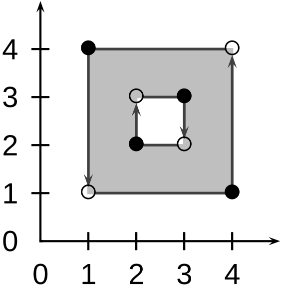

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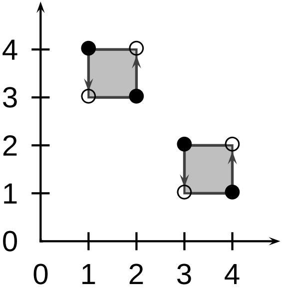

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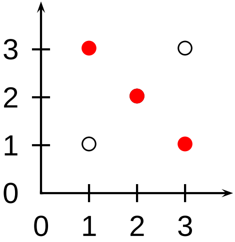

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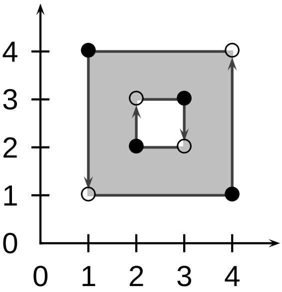

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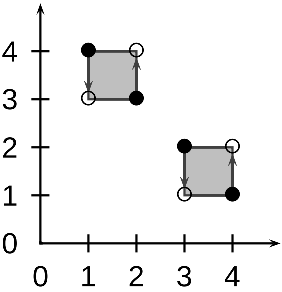

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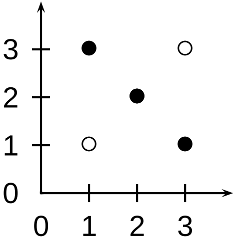

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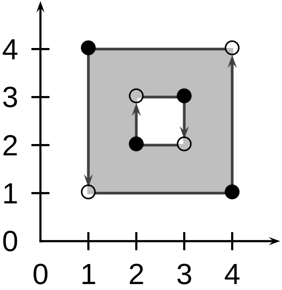

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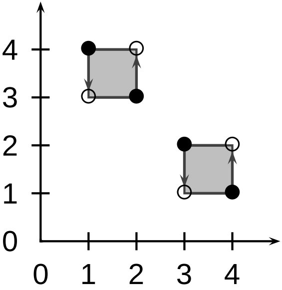

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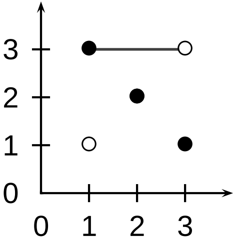

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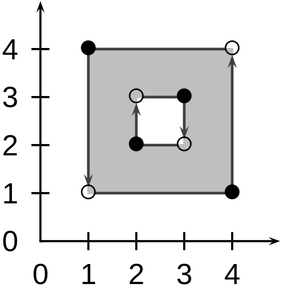

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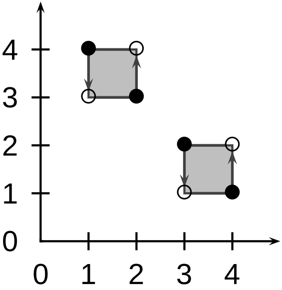

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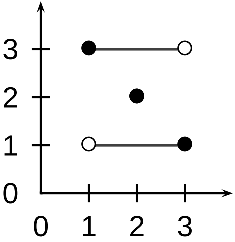

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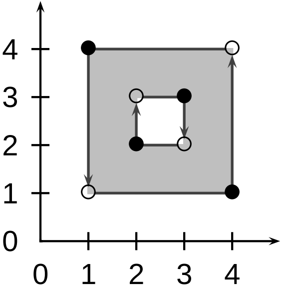

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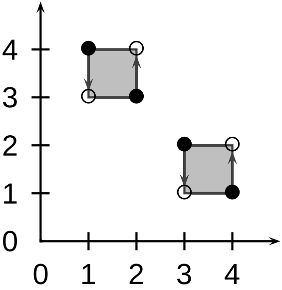

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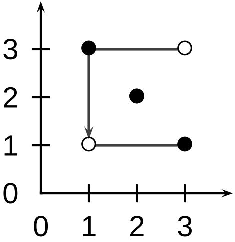

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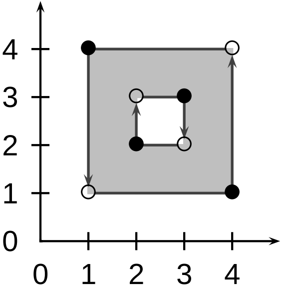

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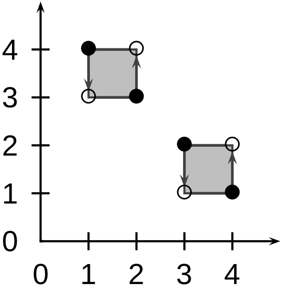

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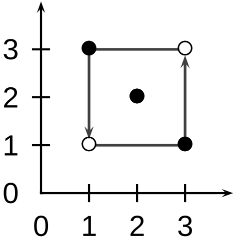
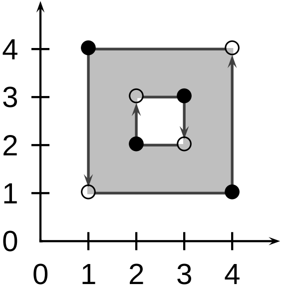
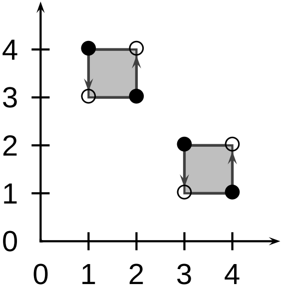


Examples of Strange Forbidden Regions

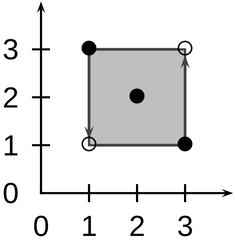
For $[1324, 4231]$:



For $[3412, 4321]$:

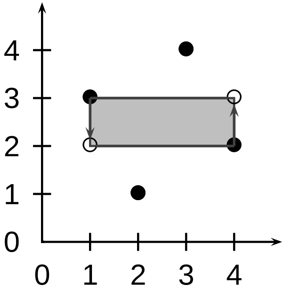


For $[123, 321]$:

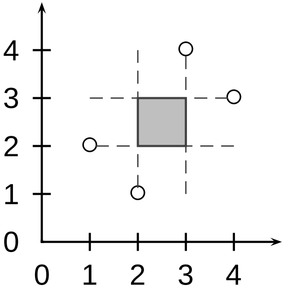


Examples of Familiar Forbidden Regions (I)

For $S_n([2143, 3142])$:



$\rightsquigarrow$

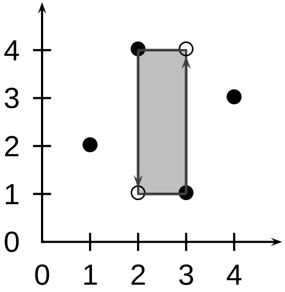


Remark. It follows that $S_n([2143, 3142]) = S_n(21\bar{3}54)$, which characterizes Planar Permutations. Similarly,

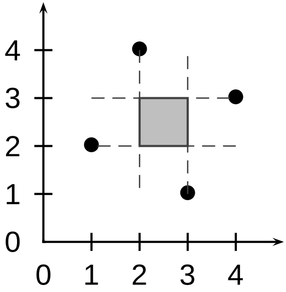
$$S_n(21\bar{3}54, 1324) = S_n([2143, 3142], [1324, 1324]).$$

Examples of Familiar Forbidden Regions (II)

For $S_n([2143, 2413])$:



$\rightsquigarrow$

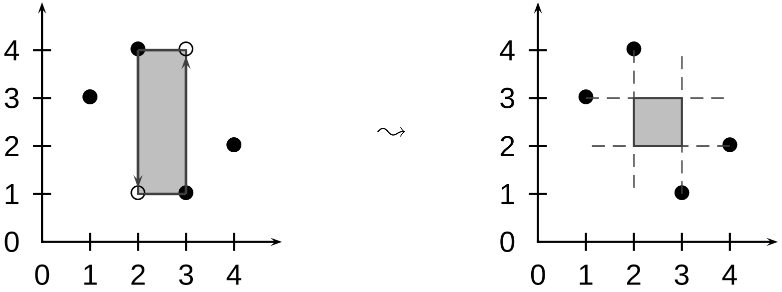


Remark. It follows that $S_n([2143, 2413]) = S_n(25\bar{3}14)$, which characterizes Baxter Permutations as

$$S_n(41\bar{3}52, 25\bar{3}14) = S_n([2143, 3142], [2143, 2413]).$$

Examples of Familiar Forbidden Regions (III)

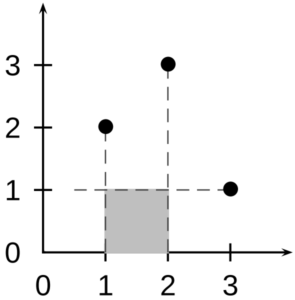
For $S_n([3142, 3412])$:



Remark. It follows that $S_n([3142, 3412]) = S_n(45\bar{3}12)$,
which characterizes Twisted Baxter Permutations as

$$S_n(45\bar{3}12, 25\bar{3}14) = S_n([3142, 3412], [2143, 2413]).$$

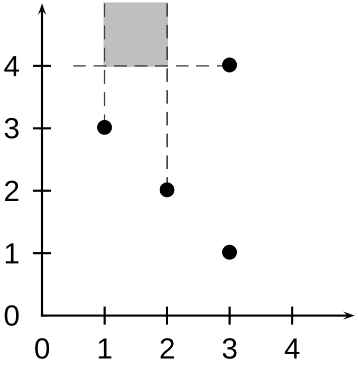
Examples of Familiar Forbidden Regions (IV)



Forbidden region for $3\bar{1}42$.

This forbidden region cannot be reduced to Interval Avoidance since it is unbounded.

Examples of Familiar Forbidden Regions (V)



Forbidden region for $3\bar{5}241$.

This forbidden region cannot be reduced to Interval Avoidance since it is unbounded.

Interval Equivalences Using Symmetries

Using inverses and reverse complements:

Interval Equivalences Using Symmetries

Using inverses and reverse complements:

$\#S_n([123,132])$

Interval Equivalences Using Symmetries

Using inverses and reverse complements:

$$\#S_n([123,132]) \stackrel{rc}{=}$$

Interval Equivalences Using Symmetries

Using inverses and reverse complements:

$$\#S_n([123,132]) \overset{rc}{=} \#S_n([123,213])$$

Interval Equivalences Using Symmetries

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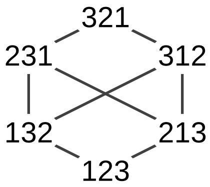

Interval Equivalences Using Symmetries

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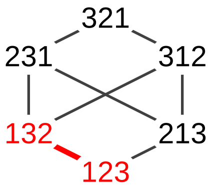

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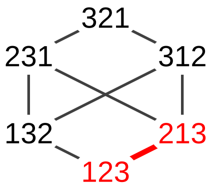

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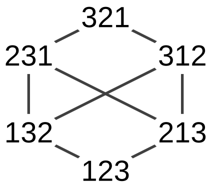

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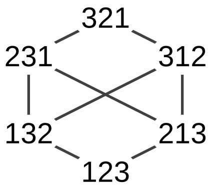

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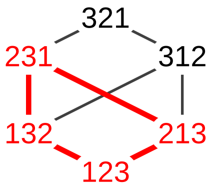

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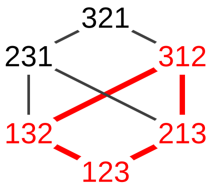

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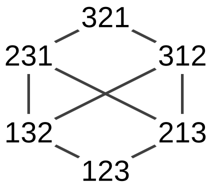

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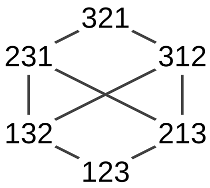

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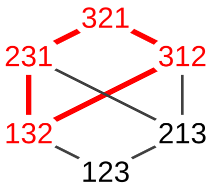

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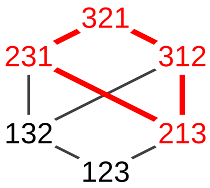

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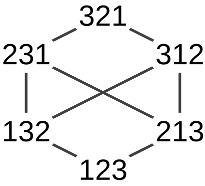

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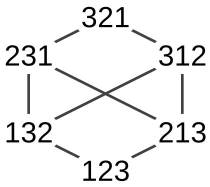

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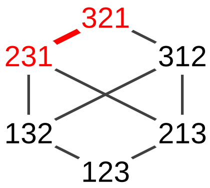

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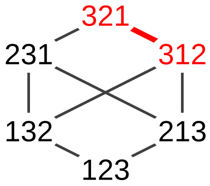

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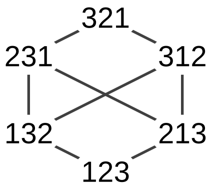

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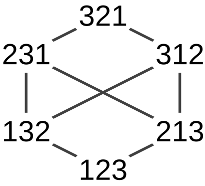

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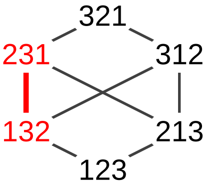

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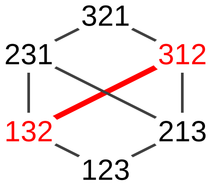

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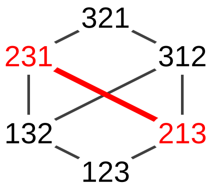

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Interval Equivalences Using Symmetries

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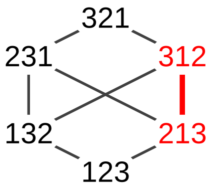

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Interval Equivalences Using Symmetries

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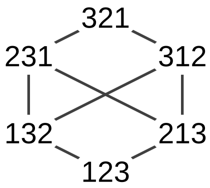

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Interval Equivalences Using Symmetries

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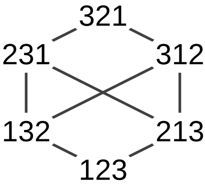

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Using reverses and complements:

Interval Equivalences Using Symmetries

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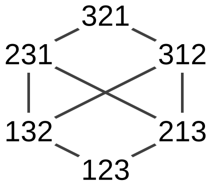
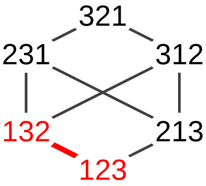


Using reverses and complements:

$$\#S_n([231,321]) \stackrel{rev}{=} \#S_n([123,132]) \stackrel{cmp}{=} \#S_n([312,321])$$

Interval Equivalences Using Symmetries

Using inverses and reverse complements:



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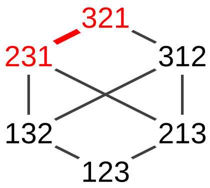

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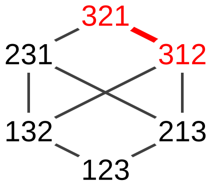
$$\#S_n([231,321]) \overset{inv}{=} \#S_n([312,321])$$

$$\#S_n([132,231]) \overset{inv}{=} \#S_n([132,312]) \overset{rc}{=} \#S_n([213,231]) \overset{inv}{=} \#S_n([213,312])$$

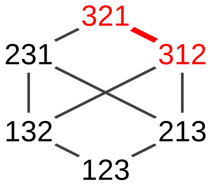


Using reverses and complements:

$$\#S_n([231,321]) \overset{rev}{=} \#S_n([123,132]) \overset{cmp}{=} \#S_n([312,321])$$

Interval Equivalences Using Symmetries

Using inverses and reverse complements:



$$\#S_n([123,132]) \stackrel{rc}{=} \#S_n([123,213])$$

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Using reverses and complements:

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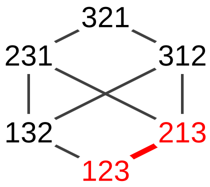
Interval Equivalences Using Symmetries

Using inverses and reverse complements:



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$$\#S_n([123,231]) \stackrel{inv}{=} \#S_n([123,312])$$

$$\#S_n([132,321]) \stackrel{rc}{=} \#S_n([213,321])$$

$$\#S_n([231,321]) \stackrel{inv}{=} \#S_n([312,321])$$

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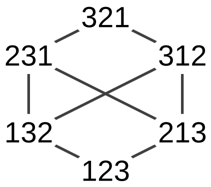
Using reverses and complements:

$$\#S_n([231,321]) \stackrel{rev}{=} \#S_n([123,132]) \stackrel{cmp}{=} \#S_n([312,321])$$

Interval Equivalences Using Symmetries

Using inverses and reverse complements:



$$\#S_n([123,132]) \overset{rc}{=} \#S_n([123,213])$$

$$\#S_n([123,231]) \overset{inv}{=} \#S_n([123,312])$$

$$\#S_n([132,321]) \overset{rc}{=} \#S_n([213,321])$$

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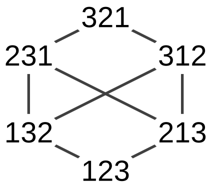
Using reverses and complements:

$$\#S_n([231,321]) \overset{rev}{=} \#S_n([123,132]) \overset{cmp}{=} \#S_n([312,321])$$

Interval Equivalences Using Symmetries

Using inverses and reverse complements:



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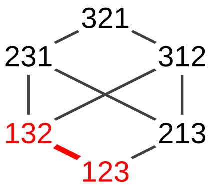
Using reverses and complements:

$$\#S_n([231,321]) \stackrel{rev}{=} \#S_n([123,132]) \stackrel{cmp}{=} \#S_n([312,321])$$

Summary (Four Distinct Symmetry Classes)

Interval Equivalences Using Symmetries

Using inverses and reverse complements:



$$\#S_n([123,132]) \overset{rc}{=} \#S_n([123,213])$$

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Using reverses and complements:

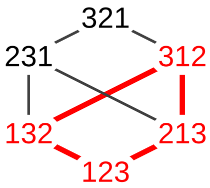
$$\#S_n([231,321]) \overset{rev}{=} \#S_n([123,132]) \overset{cmp}{=} \#S_n([312,321])$$

Summary (Four Distinct Symmetry Classes)

$S_n([123, 132])$

Interval Equivalences Using Symmetries

Using inverses and reverse complements:



$$\#S_n([123,132]) \overset{rc}{=} \#S_n([123,213])$$

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$$\#S_n([132,321]) \overset{rc}{=} \#S_n([213,321])$$

$$\#S_n([231,321]) \overset{inv}{=} \#S_n([312,321])$$

$$\#S_n([132,231]) \overset{inv}{=} \#S_n([132,312]) \overset{rc}{=} \#S_n([213,231]) \overset{inv}{=} \#S_n([213,312])$$

Using reverses and complements:

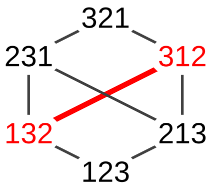
$$\#S_n([231,321]) \overset{rev}{=} \#S_n([123,132]) \overset{cmp}{=} \#S_n([312,321])$$

Summary (Four Distinct Symmetry Classes)

$S_n([123,132])$, $S_n([123,312])$

Interval Equivalences Using Symmetries

Using inverses and reverse complements:



$$\#S_n([123,132]) \overset{rc}{=} \#S_n([123,213])$$

$$\#S_n([123,231]) \overset{inv}{=} \#S_n([123,312])$$

$$\#S_n([132,321]) \overset{rc}{=} \#S_n([213,321])$$

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$$\#S_n([132,231]) \overset{inv}{=} \#S_n([132,312]) \overset{rc}{=} \#S_n([213,231]) \overset{inv}{=} \#S_n([213,312])$$

Using reverses and complements:

$$\#S_n([231,321]) \overset{rev}{=} \#S_n([123,132]) \overset{cmp}{=} \#S_n([312,321])$$

Summary (Four Distinct Symmetry Classes)

$S_n([123,132]),\ S_n([123,312]),\ S_n([132,312])$

Interval Equivalences Using Symmetries

Using inverses and reverse complements:

$$\#S_n([123,132]) \overset{rc}{=} \#S_n([123,213])$$

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$$\#S_n([132,321]) \overset{rc}{=} \#S_n([213,321])$$

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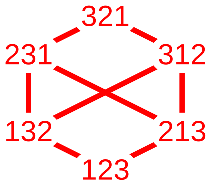
$$\#S_n([132,231]) \overset{inv}{=} \#S_n([132,312]) \overset{rc}{=} \#S_n([213,231]) \overset{inv}{=} \#S_n([213,312])$$



Using reverses and complements:

$$\#S_n([231,321]) \overset{rev}{=} \#S_n([123,132]) \overset{cmp}{=} \#S_n([312,321])$$

Summary (Four Distinct Symmetry Classes)

$S_n([123,132]),\ S_n([123,312]),\ S_n([132,312]),\ S_n([123,321])$

Main Theorem for Length Three Case

Theorem (L.-Woo)

- $S_n([123, 132]) = S_n(132)$
- $S_n([132, 312]) = S_n(312)$
- $S_n([123, 312]) = S_n(312)$

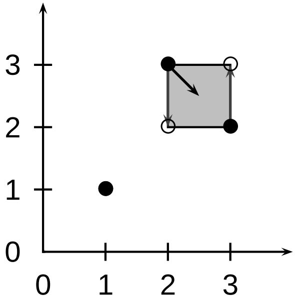
Corollary

For $\pi \leq \rho \in \mathfrak{S}_3$ and C_n the n^{th} Catalan number,

if $[\pi, \rho] \neq [123, 321]$, then

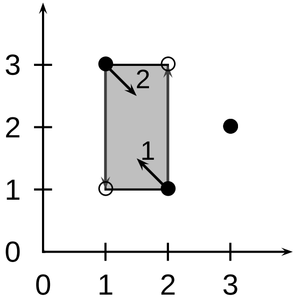
$$\#S_n([\pi, \rho]) = \#S_n(\rho) = C_n.$$

Proof for Interval Generated by $123 \leq 132$



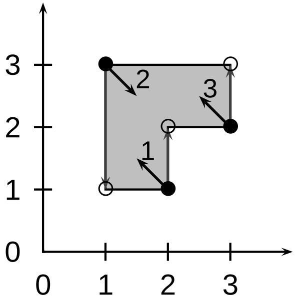
Forbidden region reduction for $[123, 132]$.

Proof for Interval Generated by $132 \leq 312$



Forbidden region reduction for $[132, 312]$.

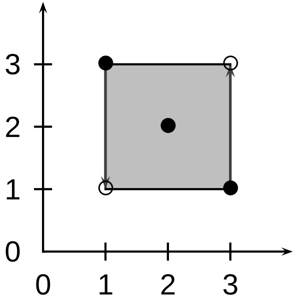
Proof for Interval Generated by $123 \leq 312$



Forbidden region reduction for $[123, 312]$.

Data for Interval Generated by $123 \leq 321$

This interval generates the following forbidden region:



The values of $\#S_n([123, 321])$ for $n = 1, 2, \ldots, 12$ are

$$1, 2, 5, 15, 51, 194, 810, 3675, 17935, 93481, 517129, 3021133.$$

Summary & Further Directions

- Interval avoidance is a very natural (and well-motivated) generalization of classical pattern avoidance.

- Intervals formed from $\mathfrak{S}_3$ all reduce to classical avoidance except for $[123, 321]$, which has so far proven elusive.

- For $n \geq 4$, "short" intervals become more subtle.

  E.g., $\tau = 53124$ contains an embedding of 4123,
  yet $\tau \in S_n([1423, 4123])$ nonetheless.

- We are also looking at what changes when Strong Bruhat Order is relaxed to Weak Bruhat Order.