1. Introduction to a new Type of Matching

2. The polynomials $U_{\Upsilon,k,n}(x)$

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Notion of a τ-match

Given $\sigma = \sigma_1 \cdots \sigma_n$, we define $red(\sigma)$ be the permutation that results by replacing the i-th largest integer that appears in the sequence σ by i.

Example: If $\sigma = 2\ 7\ 5\ 4$, then $red(\sigma) = 1\ 4\ 3\ 2$.

Given a permutation $\tau \in S_j$, we define

$$\tau\text{-}mch(\sigma) = \{i | red(\sigma_i \cdots \sigma_{i+j-1}) = \tau\}.$$

Example: If $\tau = 1\ 3\ 2\ 4$, and $\sigma = 1\ 3\ 2\ 4\ 6\ 5\ 7$, then $\tau\text{-}mch(\sigma) = \{1, 4\}$.

When $|\tau| = 2$, then $|\tau\text{-}mch(\sigma)|$ is familiar.

If $\tau = 2\ 1$, then $des(\sigma) = |\tau\text{-}mch(\sigma)|$.

If $\tau = 1\ 2$,then $rise(\sigma) = |\tau\text{-}mch(\sigma)|$.

Notion of a Υ-match and n-lap

More generally, if Υ is a set of permutations of length j, then we say that a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has a Υ match at place i provided there is a $\tau \in \Upsilon$ such that $red(\sigma_i \cdots \sigma_{i+j-1}) = \tau$.

Define Υ-$mch(\sigma)$ to be the number of Υ matches in the permutation σ.

Let τ-$nlap(\sigma)$ and $\Upsilon - nlap(\sigma)$ be the maximum number of non-overlapping τ-matches and Υ matches in σ respectively.

A more refined matching condition

Suppose we define

τ-k-$mch(\sigma) = \{i | red(\sigma_i \cdots \sigma_{i+j-1}) = red(\tau)$ and for $0 \leq s \leq j-1$, $\sigma_{i+s} = \tau_{1+s}$ mod $k\}$.

Example: If $\tau = 1\ 2$ and $\sigma = 5\ 1\ 7\ 4\ 3\ 6\ 8\ 2$, then τ-$mch(\sigma) = \{2, 5, 6\}$, but τ-2-$mch(\sigma) = \{5\}$.

Let τ-k-$emch(\sigma) = |\tau$-k-$mch(\sigma)|$.

Notion of a Υ-k-match

More generally, if Υ is a set of sequences of distinct integers of length j, then we say that a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has a Υ-k-equivalence match at place i provided there is a $\tau \in \Upsilon$ such that $red(\sigma_i \cdots \sigma_{i+j-1}) = red(\tau)$ and for all $s \in \{0, \ldots, j-1\}$, $\sigma_{i+s} = \tau_{1+s} \mod k$. Let Υ-k-$emch(\sigma)$ be the number of Υ-k-equivalence matches in the permutation σ.

We shall review the study of the polynomials

$$U_{\Upsilon,k,n}(x) = \sum_{\sigma \in S_n} x^{\Upsilon\text{-}k\text{-}emch(\sigma)} = \sum_{s=0}^{n} U^s_{\Upsilon,k,n} x^s.$$

What exactly will we study?

In particular, we shall focus on certain special cases of these polynomials where we consider only patterns of length 2.

Fix $k \geq 2$ and let A_k equal the set of all sequences we could consider for Ascents. For example, $A_4 = \{1\ 2, 1\ 3, 1\ 4, 1\ 5, 2\ 3, 2\ 4, 2\ 5, 2\ 6, 3\ 4, 3\ 5, 3\ 6, 3\ 7, 4\ 5, 4\ 6, 4\ 7, 4\ 8\}$.

Let $D_k = \{b\ a : a\ b \in A_k\}$ and $E_k = A_k \cup D_k$.

What has already been studied?

Kitaev and Remmel found explicit formulas for the coefficients $U^{s}_{\Upsilon,k,n}$ in certain special cases. In particular, they studied descents according to the equivalence class mod k of either the first or second element in a descent pair. That is, for any set $X \subseteq \{0, 1, 2, \ldots\}$, define

- $\overleftarrow{Des}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1} \ \& \ \sigma_i \in X\}$ and $\overleftarrow{des}_X(\sigma) = |\overleftarrow{Des}_X(\sigma)|$

- $\overrightarrow{Des}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1} \ \& \ \sigma_{i+1} \in X\}$ and $\overrightarrow{des}_X(\sigma) = |\overrightarrow{Des}_X(\sigma)|$

Kitaev and Remmel studied

1. $A_n^{(k)}(x) = \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{kN}(\sigma)} = \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} A_{j,n}^{(k)} x^j$ and

2. $B_n^{(k)}(x,z) = \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{kN}(\sigma)} z^{\chi(\sigma_1 \in kN)}$ $= \sum_{j=0}^{\lfloor \frac{n}{k} \rfloor} \sum_{i=0}^{1} B_{i,j,n}^{(k)} z^i x^j$.

where $kN = \{0, k, 2k, \ldots\}$.

They showed that

For all $0 \leq j \leq k-1$ and all $n \geq 0$, we have

$$\frac{A^{(k)}_{s,kn+j}}{((k-1)n+j)!} = \sum_{r=0}^{s} (-1)^{s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{s-r} \times \prod_{i=0}^{n-1} (r+1+j+(k-1)i)$$

$$= \sum_{r=0}^{n-s} (-1)^{n-s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{n-s-r} \times \prod_{i=1}^{n} (r+(k-1)i)$$

Special case

And using an identity were able to show:

$$A^{(2)}_{s,2n} = \binom{n}{s}^2 (n!)^2$$

and

$$A^{(2)}_{s,2n+1} = \frac{1}{s+1}\binom{n}{s}^2 ((n+1)!)^2$$

We will now turn to another special case of $U^s_{\Upsilon,k,n}$.

In particular, we will compute explicit formulas for $U^s_{\Upsilon,k,n}$ where

$$\Upsilon = \{(1\ k)\}.$$

Finding formulas:

Obtaining a recursion

Let

$$\Delta_{kn+j} : x^s \to sx^{s-1} + (kn+j-s)x^s$$

$$\Gamma_{kn+k} : x^s \to ((k-1)n+k+s-1)x^s+(n-s+1)x^{s+1}$$

The polynomials $U_{\{(1\ \ k)\},k,n}(x)$ satisfy the following recursions.

1. $U_{\{(1\ \ k)\},k,1}(x) = 1$,
2. For $j = 1, \ldots, k-1$, $U_{\{(1\ \ k)\},k,kn+j}(x) = \Delta_{kn+j}(U_{\{(1\ \ k)\},k,kn+j-1}(x))$, and
3. $U_{\{(1\ \ k)\},k,kn+k}(x) = \Gamma_{kn+k}(U_{\{(1\ \ k)\},k,kn+k-1}(x))$.

The basic recursions

This gives rise to the following recursions for the coefficients. For $1 \leq j \leq k-1$,

$$
\begin{aligned}
U^s_{\{(1\ \ k)\},k,kn+j} &= (kn+j-s)U^s_{\{(1\ \ k)\},k,kn+j-1} \\
&\quad +(s+1)U^{s+1}_{\{(1\ \ k)\},k,kn+j-1}
\end{aligned}
$$

and

$$
U^s_{\{(1\ \ k)\},k,kn+k} = (n-s+2)U^{s-1}_{\{(1\ \ k)\},k,kn+k-1}
$$
$$
+((k-1)n+s+k-1)U^s_{\{(1\ \ k)\},k,kn+k-1}
$$

Extreme coefficients

We have for $j = 0, \ldots, k-1$,

$$U^0_{\{(1\ k)\},k,kn+j} = ((k-1)n+j)!((k-1)n+j)^n$$

$$U^n_{\{(1\ k)\},k,kn+j} = ((k-1)n+j)!$$

Also note that

$U^m_{\{(1\ k)\},k,kn+j} = 0$ for $m > n$.

Closed form 1

Starting with the formula for $U^0_{\{(1\ \ k)\},k,kn+j}$ we can use the recursions to prove:

For all $0 \leq j \leq k-1$ and all n such that $kn+j > 0$, we have

$$U^s_{\{(1\ \ k)\},k,kn+j} =$$

$$((k-1)n+j)! \sum_{r=0}^{s} (-1)^{s-r}((k-1)n+r+j)^n \binom{(k-1)n+r+j}{r}\binom{kn+j+1}{s-r}$$

Closed form 2

Starting with the formula for $U^n_{\{(1\ \ k)\},k,kn+j}$ we can use the recursions to prove:

For all $0 \le j \le k-1$ and all n such that $kn+j > 0$, we have

$U^s_{\{(1\ \ k)\},k,kn+j} =$

$((k-1)n+j)! \sum_{r=0}^{n-s} (-1)^{n-s-r} (1+r)^n \binom{(k-1)n+r+j}{r} \binom{kn+j+1}{n-s-r}$

Karlsson-Minton hypergeometric series

A hypergeometric series is defined by

$$ {}_pF_q\left[\begin{matrix} a_1, & a_2, & \ldots, & a_p \\ b_1, & b_2, & \ldots, & b_q \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{k!(b_1)_k \ldots (b_q)_k} z^k $$

A Karlsson-Minton hypergeometric series is defined by

$$ {}_{t+1}F_t\left[\begin{matrix} c_1, & c_2, & \ldots, & c_{t+1} \\ & b_1, & \ldots, & b_t \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(c_1)_k \ldots (c_{t+1})_k}{k!(b_1)_k \ldots (b_t)_k} z^k $$

Gasper proved that

$$
{}_{t+2}F_{t+1}\left[\begin{matrix} w, & x, & b_1+d_1, & \dots & b_t+d_t \\ & x+c+1, & b_1, & \dots & b_t \end{matrix}\right] =
$$

$$
\frac{\Gamma(1+x+c)\Gamma(1-w)}{\Gamma(1+x-w)\Gamma(c+1)} \prod_{i=1}^{t} \frac{(b_i-x)_{d_i}}{(b_i)_{d_i}}
$$

$$
\times {}_{t+2}F_{t+1}\left[\begin{matrix} -c, & x, & z-b_1, & \dots & z-b_t \\ & z-w, & z-b_1-d_1, & \dots & z-b_t-d_t \end{matrix}\right]
$$

where $z = 1 + x$, $w, c, x, b_i \in \mathbb{C}$, $d_i \in \mathbb{N}$, and $\Re(c - w) > n - 1$.

A Simple Example

Let $R_n(x) = A_n^{(2)}(x) = \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} R_{s,n} x^s$. Let

$$\Delta_{2n+1} : x^s \to s x^{s-1} + (2n - s + 1)x^s$$

and

$$\Gamma_{2n+2} : x^s \to (s+1)x^s + (2n - s + 1)x^{s+1}.$$

Then Kitaev and Remmel proved the following.

The polynomials $R_n(x)_{n \geq 1}$ satisfy the following recursions.

1. $R_1(x) = 1$,

2. $R_{2n+1}(x) = \Delta_{2n+1}(R_{2n}(x))$, and

3. $R_{2n+2}(x) = \Gamma_{2n+2}(R_{2n+1}(x))$.

The basic recursions

This fact gave rise to the following recursions for $R_{s,n}(x)$.

$$\begin{aligned} R_{s,2n+1} &= (s+1)R_{s+1,2n} + (2n-s+1)R_{s,2n} \\ R_{s,2n+2} &= (s+1)R_{s,2n+1} + (2n-s+2)R_{s-1,2n+1} \end{aligned}$$

It was through these recursions that Kitaev and Remmel were able to show that

$$\begin{aligned} R_{k,2n} &= \binom{n}{k}^2 (n!)^2, \text{ and} \\ R_{k,2n+1} &= \frac{1}{k+1}\binom{n}{k}^2 ((n+1)!)^2. \end{aligned}$$

The q-recursions

To prove q-analogues of the results, let

$$\Delta^q_{2n+1} : x^s \to [s]_q \, x^{s-1} + q^s \, [2n-s+1]_q \, x^s$$

and

$$\Gamma^q_{2n+2} : x^s \to [s+1]_q \, x^s + q^{s+1} \, [2n-s+1]_q \, x^{s+1}.$$

Define $R^q_n(x)_{n\geq 1} = \sum_{s=0}^{n} R^q_{s,n} x^s$, via the following recursions.

1. $R^q_1(x,q) = 1$,

2. $R^q_{2n+1}(x,q) = \Delta^q_{2n+1}(R^q_{2n}(x))$, and

3. $R^q_{2n+2}(x,q) = \Gamma^q_{2n+2}(R^q_{2n+1}(x))$.

This fact gives rise to the following recursions for the coefficients $R^q_{s,n}(x)$.

$$\begin{aligned} R^q_{s,2n+1} &= [s+1]_q\, R^q_{s+1,2n} + q^s\, [2n-s+1]_q\, R^q_{s,2n} \\ R^q_{s,2n+2} &= [s+1]_q\, R^q_{s,2n+1} + q^s\, [2n-s+2]_q\, R^q_{s-1,2n+} \end{aligned}$$

We can then show that the solution to these recursions are

$$\begin{aligned} R^q_{k,2n} &= q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 ([n]_q!)^2, \text{ and} \\ R^q_{k,2n+1} &= \frac{q^{\binom{k+1}{2}}}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 ([n+1]_q!)^2. \end{aligned}$$

A familliar permutation statistic, maj

Given $\sigma \in S_n$, $maj(\sigma) = \sum_{i \in Des(\sigma)} i$. Foata showed that the maj statistic satisfies some simple recursions. That is, for any permutation $\tau = \tau_1 \ldots \tau_{n-1} \in S_{n-1}$, we label the spaces where we can insert n into τ to get a permutation in S_n as follows.

1. Label the space following τ_{n-1} with 0.

2. Next label the spaces that lie between descents $\tau_i > \tau_{i+1}$ from right to left with the integers $1, \ldots, des(\tau)$.

3. Finally label the remaining spaces from left to right with the integers $des(\tau) + 1, \ldots, n$.

Example: If $\tau = 3\ 9\ 2\ 8\ 5\ 4\ 1\ 6\ 7$, then spaces are labeled as follows:

$$ {}_{\overline{5}}3_{\overline{6}}9_{\overline{4}}2_{\overline{7}}8_{\overline{3}}5_{\overline{2}}4_{\overline{1}}1_{\overline{8}}6_{\overline{9}}7_{\overline{0}}. $$

Then Foata proved that if $\tau^{(i)}$ is the result of inserting n into the space labeled i, then for all $i \in \{0, \ldots, n\}$,

$$ maj(\tau^{(i)}) = i + maj(\tau). $$

The natural q-statistic

We can use a similar labeling procedure to define a statistic $Emaj$ such that $R_n^q(x,q) = \sum_{\sigma \in S_n} q^{Emaj(\sigma)} x^{\overleftarrow{des}_E}$.

Look at the operators that produced the recursions.

$$\Delta_{2n+1}^q : x^s \to [s]_q \, x^{s-1} + q^s \, [2n - s + 1]_q \, x^s$$

and

$$\Gamma_{2n+2}^q : x^s \to [s+1]_q \, x^s + q^{s+1} \, [2n - s + 1]_q \, x^{s+1}.$$

Example: $\sigma = 3\ 9\ 2\ 8\ 5\ 4\ 1\ 6\ 7$. Then the E-canonical labeling of σ is

$$_{\overline{3}}3_{\overline{4}}9_{\overline{5}}2_{\overline{6}}8_{\overline{2}}5_{\overline{7}}4_{\overline{1}}1_{\overline{8}}6_{\overline{9}}7_{\overline{0}}.$$

A more general example

Let us examine the q-analogue of the polynomials $U_{\{(1k)\},k,n}(x)$. For $j = 0, \ldots, k-2$ let

$$\Delta^q_{kn+j} : x^s \to [s]_q x^{s-1} + q^s [kn + j - s]_q x^s$$

and

$$\Gamma^q_{kn+k} : x^s \to [(k-1)n + k + s - 1]_q x^s$$

$$+ q^{(k-1)n+k+s-1}[n - s + 1] x^{s+1}.$$ Define:

1. $U^q_{\{(1k)\},k,1}(x,q) = 1$,

2. For $j = 1, \ldots, k-1$, $U^q_{\{(1k)\},k,kn+j}(x,q) = \Delta^q_{kn+j}(U^q_{\{(1k)\},k,kn+j-1}(x,q))$, and

3. $U^q_{\{(1k)\},k,kn+k}(x,q) = \Gamma^q_{kn+k}(U^q_{\{(1k)\},k,kn+k-1}(x,q))$.

The q-formulas

If $U^{q}_{\{(1k)\},k,kn+j}(x,q) = \sum_{s=0}^{n} U^{s,q}_{\{(1k)\},k,kn+j} x^s$, then

For all $0 \le j \le k-1$ and all n such that $kn+j > 0$, we have

$$\frac{U^{s,q}_{\{(1k)\},k,kn+j}}{[(k-1)n+j]_q!}$$
$$= \sum_{r=0}^{s} (-1)^{s-r} q^{\binom{s}{2}-\binom{r}{2}-r(s-r)}$$
$$\times [(k-1)n+j+r]_q^n \begin{bmatrix} (k-1)n+j+r \\ r \end{bmatrix}_q \begin{bmatrix} kn+j+1 \\ s-r \end{bmatrix}_q$$

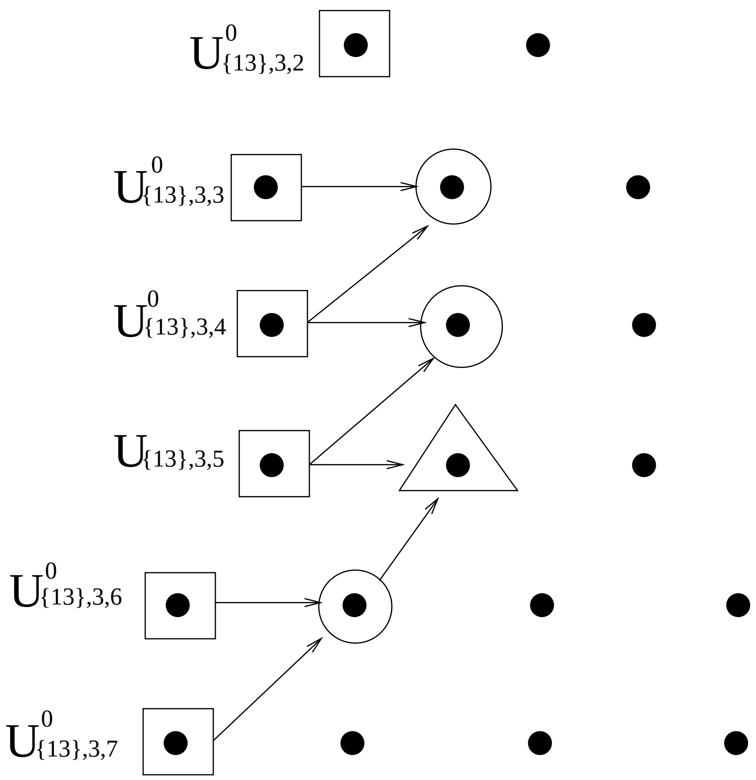
$$= \sum_{r=0}^{n-s} (-1)^{n-s-r} q^{\binom{n-s}{2}-\binom{r}{2}-r(n-s-r)-\binom{n+1}{2}+s(kn+j)}$$
$$\times [1+r]_q^n \begin{bmatrix} (k-1)n+j+r \\ r \end{bmatrix}_q \begin{bmatrix} kn+j+1 \\ n-s-r \end{bmatrix}_q$$

Some differences in the q-case

When $q = 1$ we know that

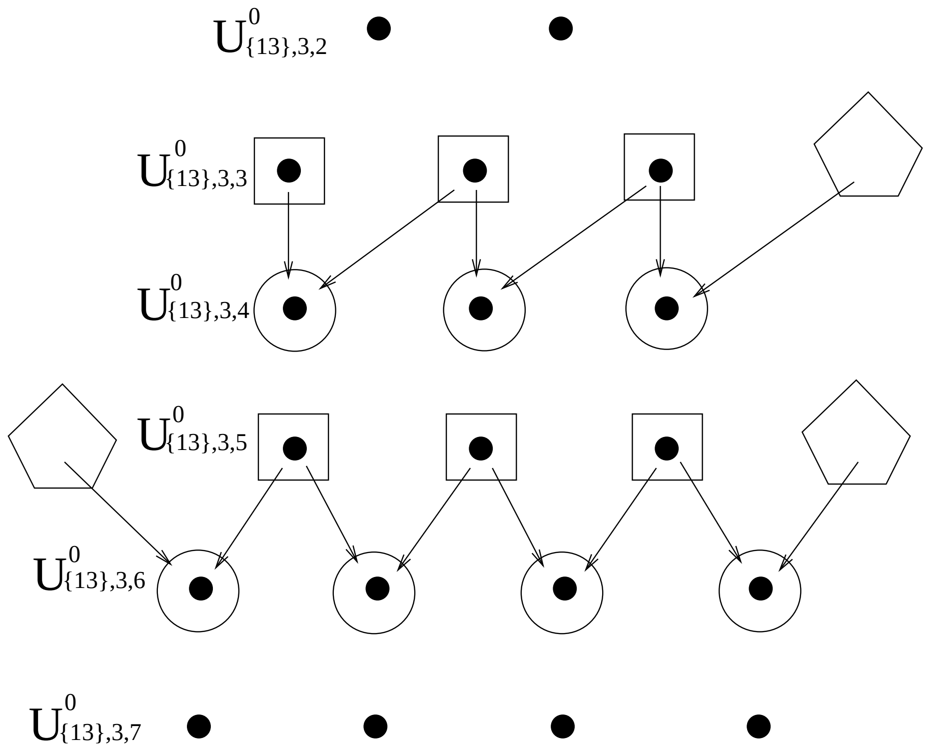
$$U^0_{\{(1k)\},k,kn+j} = ((k-1)n+j)!\,((k-1)n+j)^n \text{ and}$$
$$U^n_{\{(1k)\},k,kn+j} = ((k-1)n+j)!$$

and we used these facts to prove the formulas hold.



Some differences in the q-case

In the general q-case, we need more. $U^{-1,q}_{\{(1k)\},k,kn+j}$ and $U^{n+1,q}_{\{(1k)\},k,kn+j}$ make sense but must be 0 by our definitions.



This can be proven with the following Theorem.

A necessary Theorem

For all positive integers $k, n, j, z_1, \ldots, z_n$ and any function $\theta(r)$ where $kn + j > 0$, $0 < z_i < (k-1)n + j$, and $\theta(r+1) = \theta(r) - (n-r)$,

$$\sum_{r=0}^{n+1} (-1)^{n+1-r} q^{\theta(r)} \prod_{i=1}^{n} [z_i + r]_q \times \begin{bmatrix} kn+j+1 \\ (k-1)n+j, r, n+1-r \end{bmatrix}_q = 0.$$

An alternate formula

It has also been shown that

$$U^{s}_{\{(1k)\},k,kn+j} = \sum_{r=s}^{n} (-1)^{r-s} \binom{r}{s} (kn+j-r)! S_{n+1,n+1-r}$$

where $S_{n,k}$ is the Stirling number of the second kind, i.e. $S_{n,k}$ is the number of set partitions of $\{1,\ldots,n\}$ into k parts.

Conjecture: $U^{s,q}_{\{(1k)\},k,kn+j} =$
$\sum_{r=s}^{n} (-1)^{r-s} q^{\binom{n+1-s}{2}+r((k-1)n+j)-\binom{n+1}{2}+ns} \begin{bmatrix} r \\ s \end{bmatrix}_q$
$\times [kn+j-r]_q! S^{q}_{n+1,n+1-r}$ where $S^{q}_{n,k}$ is the q-Stirling number of the second kind that defined by the following recursion.

$$S^{q}_{0,0} = 1, S^{q}_{n,k} = 0 \text{ if } k<0 \text{ or } k>n, \text{ and,}$$
$$S^{q}_{n,k} = S^{q}_{n-1,k-1} + [k]_q \, S^{q}_{n-1,k} \text{ if } 0 \le k \le n.$$

Unanswered Questions

1. General Subset Problem

2. PQ-Analogues of U-Coefficients

3. Consider Matching where $|\tau| > 2$

4. Find a Combinatorial Interpretation of the Closed Forms

5. Wilf Equivalence Classes