Counting Consecutive Pattern-Avoiding Permutations with Perron and Frobenius

Richard Ehrenborg, University of Kentucky
Sergey Kitaev, Reykjavik University
Peter Perry, University of Kentucky

Contents

Peter, you are very wise to go into discrete mathematics. The real number line was invented by dead white males.

Maciej Zworski

1. Consecutive Pattern-Avoiding Permutations

Let $\mathfrak{S}_n$ be the group of permutations on n symbols. Write $\pi \in \mathfrak{S}_n$ as

$$\pi = (\pi_1, \pi_2, \ldots, \pi_n)$$

where the π_j are integers

A permutation $\pi \in \mathfrak{S}_n$ is 123-*avoiding* if there is no integer k with $1 \leq k \leq n-2$ and $\pi_k < \pi_{k+1} < \pi_{k+2}$

Let α_n be the number of such permutations in $\mathfrak{S}_n$

Problem Find the asymptotics of α_n as $n \to \infty$.

Solution The asymptotic formula

$$\frac{\alpha_n(123)}{n!} = \lambda_0^{n+1} \exp\left(\frac{1}{2\lambda_0}\right) + \mathcal{O}\left(\lambda_{-1}^{n-1}\right)$$

holds where

$$\lambda_k = \frac{\sqrt{3}}{2\pi(k+1/3)}$$

Remark The leading asymptotics were obtained earlier by Elizalde and Noy (2003)

We'll discuss the analysis of 123-avoiding permutations via the spectral theory of integral operators. The method applies to a wide range of counting problems involving consecutive pattern-avoiding permutations, and gives detailed asymptotic expansions in some cases of interest.

2. An Integral Operator Related to the Counting Problem

There is a one-to-one correspondence between permutations of $\mathfrak{S}_n$ and simplices in the standard triangulation of $[0,1]^n$

"Forbidden" permutations correspond to simplices whose points $x = (x_1, \ldots, x_n)$ in $[0,1]^n$ have $x_j < x_{j+1} < x_{j+2}$ for some j, $1 \le j \le n-2$.

"Allowed" (i.e., 123-avoiding) permutations in $\mathfrak{S}_n$ correspond to simplices S in $[0,1]^n$ for which no such points occur.

We will use this observation to pose the counting problem in terms of an integral operator acting on functions on $[0,1]^2$.

For $x \in [0,1]^3$, let

$$\chi_3(x_1, x_2, x_3) = \begin{cases} 0 & \text{if } x_1 \le x_2 \le x_3 \\ 1 & \text{otherwise} \end{cases}$$

and for $n \ge 4$ let

$$\chi_n(x_1, \ldots, x_n) = \prod_{j=1}^{n-2} \chi_3(x_j, x_{j+1}, x_{j+2})$$

Thus χ_n is a characteristic function for simplices in $[0,1]^n$ corresponding to allowed permutations.

It follows that

$$\int_{[0,1]^n} \chi_n(x)\, dx = \frac{\alpha_n}{n!}$$

The Integral Operator

Define a linear mapping from functions on $[0,1]^2$ into themselves by

$$(Tf)(x_1,x_2) = \int_0^1 \chi_3(t,x_1,x_2) f(t,x_1)\, dt$$

The mapping T is *positivity preserving*, i.e., if $f(x) \geq 0$ for all x, then $(Tf)(x) \geq 0$ for all x as well.

We will see that T (usually) has a positive eigenvalue of greatest modulus that determines the leading asymptotics of α_n as $n \to \infty$.

Let $\mathbf{1}$ denote the function on $[0,1]^2$ with constant value 1.

Note that

$$\begin{aligned} T(\mathbf{1})(x) &= \int_0^1 \chi_3(t_1, x_1, x_2) dt_1 \\ T^2(\mathbf{1})(x) &= \int_0^1 \chi_3(t_2, x_1, x_2) \int_0^1 \chi_3(t_1, t_2, x_1)\, dt_1\, dt_2 \end{aligned}$$

so, inductively

$$T^k(\mathbf{1})(x_1, x_2) =$$

$$\int_0^1 \chi_3(t_1, t_2, t_3) \chi_3(t_2, t_3, t_4) \ldots \chi_3(t_k, x_1, x_2)\, dt_1 \cdots dt_k$$

Hence

$$\frac{\alpha_{k+2}}{(k+2)!} = \int_{[0,1]^2} (T^k \mathbf{1})(x_1, x_2)\, dx_1\, dx_2$$

Recall inner product for functions on $[0,1]^2$:

$$\langle f, g \rangle = \int_{[0,1]^2} f(x_1, x_2) g(x_1, x_2)\, dx_1\, dx_2$$

Then

$$\frac{\alpha_{k+2}}{(k+2)!} = \left\langle \mathbf{1}, T^k \mathbf{1} \right\rangle$$

Generalization

Suppose

- $S \subset \mathfrak{S}_{m+1}$ is a consecutive pattern of length $(m+1)$
- $\alpha_n(S)$ is the number of S-avoiding permutations in $\mathfrak{S}_n$
- $\chi_S(x_1, \ldots, x_{m+1})$ is the characteristic function of simplices in $[0,1]^{m+1}$ corresponding to allowed permutations in $\mathfrak{S}_{m+1}$

Define:

$$(T_S f)(x_1, \ldots, x_m) =$$

$$\int_0^1 \chi_S(t, x_1, \ldots, x_m) f(t, x_1, \ldots, x_{m-1})\, dx_1 \cdots dx_m$$

Then:

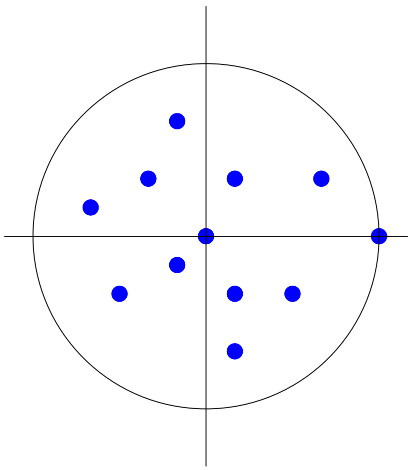
$$\alpha_{k+m}(S) = \left\langle \mathbf{1}, T_S^k \mathbf{1} \right\rangle$$

The behavior of powers T^k is governed by the eigenvalues of T. The largest eigenvalue of T determines the asymptotics of α_k.

3. The Perron-Frobenius and Krein-Rutman Theorem

For a real $m \times m$ matrix A with eigenvalues $\lambda_1, \ldots, \lambda_m$, the *spectral radius* of A is

$$r(A) = \sup_{1 \le i \le m} |\lambda_i|.$$

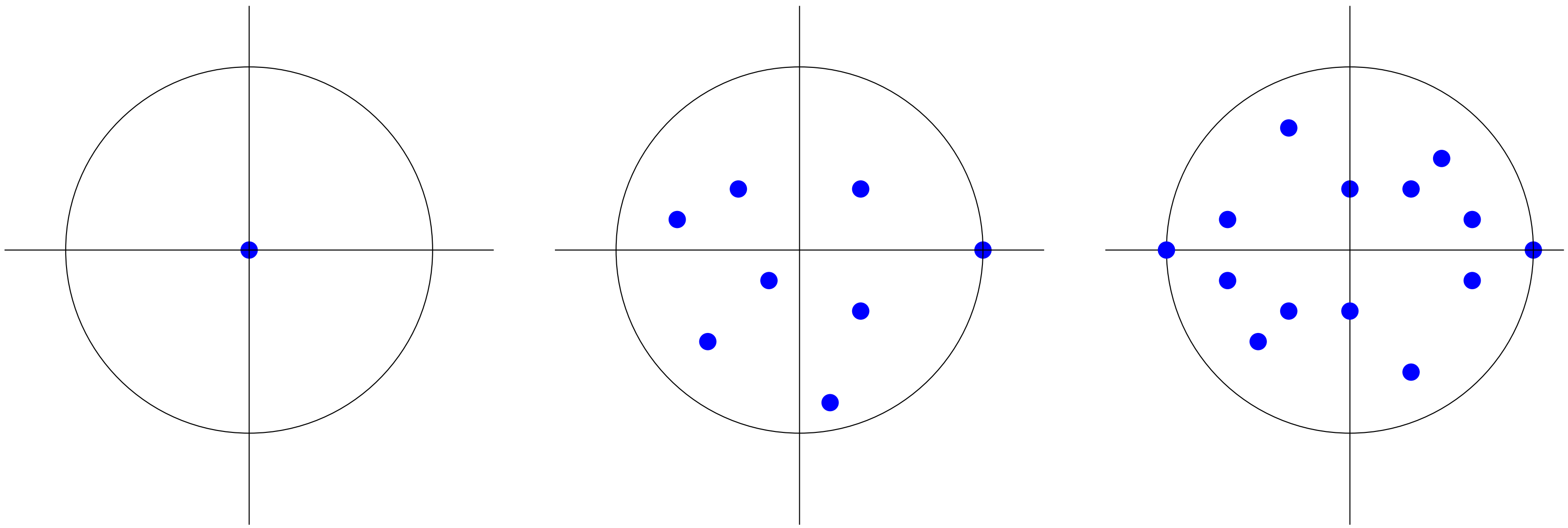


The Spectral Radius

Theorem (Perron-Frobenius) *Suppose that A is a nonzero matrix with nonnegative entries. Let $\rho = r(A)$. Either:*
(a) $\rho = 0$ and A is nilpotent, or
(b) $\rho > 0$, and ρ is an eigenvalue of A with nonzero, nonnegative eigenvector v. In this case, all of the eigenvalues λ with $|\lambda| = \rho$ take the form $\lambda = \zeta\rho$ where ζ is a root of unity.

Note that A^* also satisfies the hypothesis so, in the second case, A^* has eigenvalue ρ and a nonnegative eigenvector w as well.

Three Cases of Perron-Frobenius



ρ(A)=0 ρ(A) nonzero ρ(A) nonzero

Let

$$\langle u, v \rangle = \sum_{i=1}^{m} \overline{u}_i v_j$$

and

$$\mathbf{1} = (1, 1, \ldots, 1)$$

Suppose A is a nonzero matrix with nonnegative entries. Denote by ρ the spectral radius of A.

Consider

$$r_n = \langle \mathbf{1}, A^n \mathbf{1} \rangle$$

Either:

(a)There is an N so $r_n = 0$ for $n \geq N$, or

(b) $r_n > 0$ for all n and

$$\lim_{n\to\infty} \left(r_n^{1/n} \right) = \rho$$

In the second case, if $\lambda = \rho$ is the only eigenvalue of modulus ρ, then

$$r_n = c\rho^n + \mathcal{O}(\rho_1^n)$$

where $\rho_1 < \rho$ and

$$c = \langle w, \mathbf{1} \rangle \langle \mathbf{1}, v \rangle$$

Here $Av = \rho v$ and $A^* w = \rho w$. We normalize so $\langle v, w \rangle = 1$

Hints for the proof: If

$$Av_k = \lambda_k v_k, \quad A^* w_k = \overline{\lambda_k} w_k$$

where

$$\langle w_j, v_k \rangle = \delta_{jk}$$

then

$$A^n x = \sum_{k=1}^{m} \lambda_k^n \langle w_k, x \rangle v_k$$

so

$$\langle \mathbf{1}, A^n \mathbf{1} \rangle = \sum_{k=1}^{m} \lambda_k^n \langle w_k, \mathbf{1} \rangle \langle \mathbf{1}, v_k \rangle$$

The leading terms correspond to those λ_k of maximum modulus

These terms sum to $\rho^n f(n)$ where f is strictly positive and periodic in n

Linear Operators

Definition A linear operator T on functions is *positivity preserving* if $Tf(x) \geq 0$ whenever $f(x) \geq 0$ and *positivity improving* if $(Tf)(x) > 0$ strictly if $f(x) \geq 0$ and f is nonzero.

Theorem (Krein-Rutman 1948) *If T is positivity preserving and compact, then either:*
(a) T has zero spectral radius, or
(b) T has nonzero spectral radius ρ, and there is a nonzero nonnegative function v so that $Tv = \rho v$.
In the second case, if T is positivity improving, then ρ is the unique eigenvalue of maximal modulus, and all other eigenvalues of T satisfy $|\lambda| \leq \rho_1$ for $0 \leq \rho_1 < \rho$.

4. Asymptotics

Recall that for a pattern S of length $(m+1)$,

$$\frac{\alpha_n}{n!} = \langle \mathbf{1}, T_S^{n-m} \mathbf{1} \rangle$$

$\rho(S)$ is the spectral radius of T_S

Theorem *Suppose that S is a nonempty pattern. Then*

$$\rho(S) = \lim_{n \to \infty} \left(\alpha_n(S)/n! \right)^{1/n}$$

Either $\rho(S) = 0$ or $\rho(S) > 0$!

Later, we will describe a combinatorial condition which guarantees that $\rho(T_S) > 0$.

Example 1 Suppose $S = \{132, 231\}$. An S-avoiding permutation has "no peaks" and one can show that $\alpha_n(S) = 2^{n-1}$. Thus $\rho(S) = 0$.

Example 2 Suppose that $S = \{123, 321\}$. Then $\alpha_n(S) = 2E_n$ where E_n is the nth Euler number. T_S has eigenvalues $\pm 2/\pi$ of maximum modulus and the spectrum is invariant under $\lambda \mapsto -\lambda$. There is a complete asymptotic expansion for $\alpha_n(S)$

Example 3 Suppose that $S = \{123\}$. Then T_S has a maximal positive eigenvalue $3\sqrt{3}/(2\pi)$ and all other eigenvalues are real and of smaller modulus. There is a complete asymptotic expansion for $\alpha_n(S)$:

$$\lambda_k = \frac{\sqrt{3}}{2\pi(k + 1/3)}$$

There is an infinite graph H_S associated with the pattern S which is essentially an infinite de Brujin graph with certain edges removed. For patterns of length $m+1$:

- Its vertices are interior points of simplices of the unit m-cube

- Two vertices x and y are connected if $x_1 \neq y_m$, $x_{j+1} = y_j$, and $x_1 x_2 \cdots x_m y_m$ is order equivalent to an allowed permutation

Theorem $\rho(S) > 0$ *if and only if* H_S *has a directed cycle*

We can also give conditions on H_S under which $\rho(S)$ the unique eigenvalue of maximum modulus

5. Further Remarks

If S is a consecutive pattern of length $m+1$ we have

$$\frac{\alpha_{k+m}(S)}{(k+m)!} = \left\langle \mathbf{1}, T^k \mathbf{1} \right\rangle$$

It follows that

$$\sum_{n=0}^{\infty} \alpha_n(S) \frac{z^n}{n!} =$$

$$1 + \cdots + z^m + z^{m+1} \left\langle \mathbf{1}, (I - zT_S)^{-1} T_S \mathbf{1} \right\rangle$$

Thus the radius of convergence of the generating function is determined by the spectrum of T_S.

Krein-Rutman's theorems imply that

$$T_S^n = \rho(S)U^n + V^n$$

where U is a permutation matrix and V is "negligible"

Question How is the permutation related to S?

Question What can be said about $\alpha_n(S)$ when $\rho(S) = 0$?