Permutations and words counted by consecutive patterns

Jeff Remmel
Department of Mathematics and Computer Science
University of California, San Diego

Joint work with Tony Mendes, Department of Mathematics,
California Polytechnic State University

$\sigma = \sigma_1 \ldots \sigma_n \in S_n$: assume $\sigma_{n+1} = n+1$

$$Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\} \qquad Rise(\sigma) = \{i : \sigma_i < \sigma_{i+1}\}$$

$$des(\sigma) = |Des(\sigma)| \qquad rise(\sigma) = |Rise(\sigma)|$$

$$maj(\sigma) = \sum_{i \in Des(\sigma)} i \qquad comaj(\sigma) = \sum_{i \in Rise(\sigma)} i$$

$$rlmaj(\sigma) = \sum_{i \in Des(\sigma)} n - i \qquad rlcomaj(\sigma) = \sum_{i \in Rise(\sigma)} n - i$$

$$inv(\sigma) = \sum_{i<j} \chi(\sigma_i > \sigma_j) \qquad coinv(\sigma) = \sum_{i<j} \chi(\sigma_i < \sigma_j)$$

$$exc(\sigma) = |\{i : i < \sigma_i\}| \qquad dec(\sigma) = |\{i : i > \sigma_i\}|$$

where for any statement A, $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false. Also if $\alpha^1, \ldots, \alpha^k \in S_n$, then we shall write $comdes(\alpha^1, \ldots, \alpha^k) = |\bigcap_{i=1}^k Des(\alpha^i)|$.

$$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$$

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

$$\begin{bmatrix} n \\ \lambda_1, \ldots, \lambda_\ell \end{bmatrix}_q = \frac{[n]_q!}{[\lambda_1]_q! \cdots [\lambda_\ell]_q!}$$

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + \cdots + p^1 q^{n-2} + q^{n-1} = \frac{p^n - q^n}{p-q}.$$

1) $\sum_{n=0}^{\infty} \frac{u^n}{n!} \sum_{\sigma \in S_n} x^{des(\sigma)} = \frac{1-x}{-x+e^{u(x-1)}}$

(2) (Carlitz 1970)
$\sum_{n=0}^{\infty} \frac{u^n}{(n!)^2} \sum_{(\sigma,\tau) \in S_n \times S_n} x^{comdes(\sigma,\tau)} = \frac{1-x}{-x+J(u(x-1))}$.

3) (Stanley 1979) $\sum_{n=0}^{\infty} \frac{u^n}{[n]!} \sum_{\sigma \in S_n} x^{des(\sigma)} q^{inv(\sigma)} = \frac{1-x}{-x+e_q(u(x-1))}$.

4) (Stanley 1979) $\sum_{n=0}^{\infty} \frac{u^n}{[n]!} \sum_{\sigma \in S_n} x^{des(\sigma)} q^{coinv(\sigma)} = \frac{1-x}{-x+E_q(u(x-1))}$.

5) (Fedou and Rawlings 1995)
$\sum_{n=0}^{\infty} \frac{u^n}{[n]_q![n]_p!} \sum_{(\sigma,\tau) \in S_n \times S_n} x^{comdes(\sigma,\tau)} q^{inv(\sigma)} p^{inv(\tau)} = \frac{1-x}{-x+J_{q,p}(u(x-1))}$.

$J(u) = \sum_{n \geq 0} \frac{u^n}{n!n!}$, $\qquad e_q(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q!} q^{\binom{n}{2}}$,

$E_q(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q!}$, and $\qquad J_{q,p}(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q![n]_p!} q^{\binom{n}{2}} p^{\binom{n}{2}}$.

6) Foata-Han (1997)

Let $(x,q)_0 = 1$ and $(x,q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x)$ for $n > 0$.

$$C_n(z,x,q,y,p) = \sum_{(\sigma,\tau)\in S_n\times S_n} z^{comdes(\sigma^{-1},\tau^{-1})} x^{des(\sigma)} q^{rlmaj(\sigma)} y^{rise(\tau)} p^{rlcomaj(\tau)}.$$

$$\sum_{n\geq 0} t^n \frac{C_n(z,x,q,y,p)}{(x,q)_{n+1}(y,p)_{n+1}} = \sum_{i,j\geq 0} \frac{x^i y^j}{1+\sum_{n\geq 1}(t(z-1))^{n-1} \begin{bmatrix} i+1 \\ n \end{bmatrix}_q \begin{bmatrix} j+n \\ n \end{bmatrix}_p}.$$

7) Remmel-Mendes (2004)

$R_n(z,x,q,y,p,Q,P) =$
$\sum_{(\alpha,\beta,\gamma,\delta)\in S_n^4} z^{comdes(\alpha^{-1},\beta^{-1},\gamma,\delta)} x^{des(\alpha)} q^{rlmaj(\alpha)} y^{rise(\beta)} p^{rlcomaj(\beta)} Q^{inv(\gamma)} P^{coinv(\delta)}$
and set

$$F^{i,j}(t,q,p,Q,P) = \sum_{n\geq 0} t^n \frac{q^{\binom{n}{2}} Q^{\binom{n}{2}} \begin{bmatrix} i+1 \\ n \end{bmatrix}_q \begin{bmatrix} j+n \\ n \end{bmatrix}_p}{[n]_Q![n]_P!}.$$

Then we can use the combinatorial mechanism described above with 4-tuples of permutations instead of pairs of permutations to prove that

$$\sum_{n\geq 0} \frac{R_n(z,x,q,y,p,Q,P)t^n}{[n]_Q![n]_P!(x,q)_{n+1}(y,p)_{n+1}} = \sum_{i,j\geq 0} x^i y^j \frac{1-t}{-t+F^{i,j}(t(z-1),q,p,Q,P)}.$$

Extensions of the Previous Generating functions

Fix a finite set $S \subseteq \{1, 2, \ldots\}$.

1) $\sum_{n=0}^{\infty} \frac{u^n}{n!} \sum_{\sigma \in S_n, S \subseteq Des(\sigma)} x^{des(\sigma)}$

2) $\sum_{n=0}^{\infty} \frac{u^n}{(n!)^2} \sum_{(\sigma,\tau) \in S_n \times S_n, S \subseteq Comdes(\sigma,\tau)} x^{comdes(\sigma,\tau)}$

3) $\sum_{n=0}^{\infty} \frac{u^n}{[n]_q!} \sum_{\sigma \in S_n, S \subseteq Des(\sigma)} x^{des(\sigma)} q^{inv(\sigma)}$.

4) $\sum_{n=0}^{\infty} \frac{u^n}{[n]_q!} \sum_{\sigma \in S_n, S \subseteq Des(\sigma)} x^{des(\sigma)} q^{coinv(\sigma)}$.

5) $\sum_{n=0}^{\infty} \frac{u^n}{[n]_q![n]_p!} \sum_{(\sigma,\tau) \in S_n \times S_n, S \subseteq Comdes(\sigma,\tau)} x^{comdes(\sigma,\tau)} q^{inv(\sigma)} p^{inv(\tau)}$.

Elementary & homogeneous symmetric functions

The n^{th} elementary symmetric function e_n is defined by

$$\sum_{n\geq 0} e_n t^n = \prod_i (1 + x_i t).$$

The n^{th} homogeneous symmetric function h_n is defined by

$$\sum_{n\geq 0} h_n t^n = \prod_i \frac{1}{1 - x_i t}.$$

The n^{th} power symmetric function p_n is defined by

$$p_n = \sum_i x_i^n.$$

If $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a partition, then
$h_\lambda = \prod_{i=1}^k h_{\lambda_i}$, $e_\lambda = \prod_{i=1}^k e_{\lambda_i}$, and $p_\lambda = \prod_{i=1}^k p_{\lambda_i}$,

$$
\begin{aligned}
1 &= \left(\prod_i \frac{1}{1-x_i t}\right)\left(\prod_i (1-x_i t)\right) \\
&= \left(\sum_{n\geq 0} h_n t^n\right)\left(\sum_{n\geq 0} e_n (-t)^n\right)
\end{aligned}
$$

$$
\sum_{n\geq 0}\left(\sum_{i=0}^{n} (-1)^i e_i h_{n-i}\right) t^n = 1
$$

$\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0$ for $n \geq 1$.

$\sum_{n\geq 0} h_n t^n = \frac{1}{\left(\sum_{n\geq 0} e_n (-t)^n\right)}$.

Brenti(1993) Define a ring homomorphism $\xi : \Lambda \to Q[x]$ by setting

$$\xi(e_k) = \frac{(x-1)^{k-1}}{k!}$$

where e_k is the k-th elementary symmetric function and $\xi(e_0) = 1$.

$$n!\xi(h_n) = \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} \text{ and } \frac{n!}{z_\lambda}\xi(p_\lambda) = \sum_{\sigma \in S_n(\lambda)} x^{\mathrm{exc}(\sigma)} \qquad (1)$$

where if $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ is a partition of n, then $S_n(\lambda)$ is the set of permutations in S_n with cycle type λ, and $z_\lambda = \prod_{i=1}^{n} i^{m_i} m_i!$.

Let $\overline{x} = (x_1, x_2, \ldots)$ and $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$,

$$\begin{aligned} h_n(\overline{x}) &= \sum_{1 \leq i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n} & h_\lambda(\overline{x}) = h_{\lambda_1}(\overline{x}) \cdots h_{\lambda_\ell}(\overline{x}) \\ e_n(\overline{x}) &= \sum_{1 \leq i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n} & e_\lambda(\overline{x}) = e_{\lambda_1}(\overline{x}) \cdots e_{\lambda_\ell}(\overline{x}) \\ p_n(\overline{x}) &= \sum_i x_i^n & p_\lambda(\overline{x}) = p_{\lambda_1}(\overline{x}) \cdots p_{\lambda_\ell}(\overline{x}) \end{aligned}$$

We are interested on the expansions.

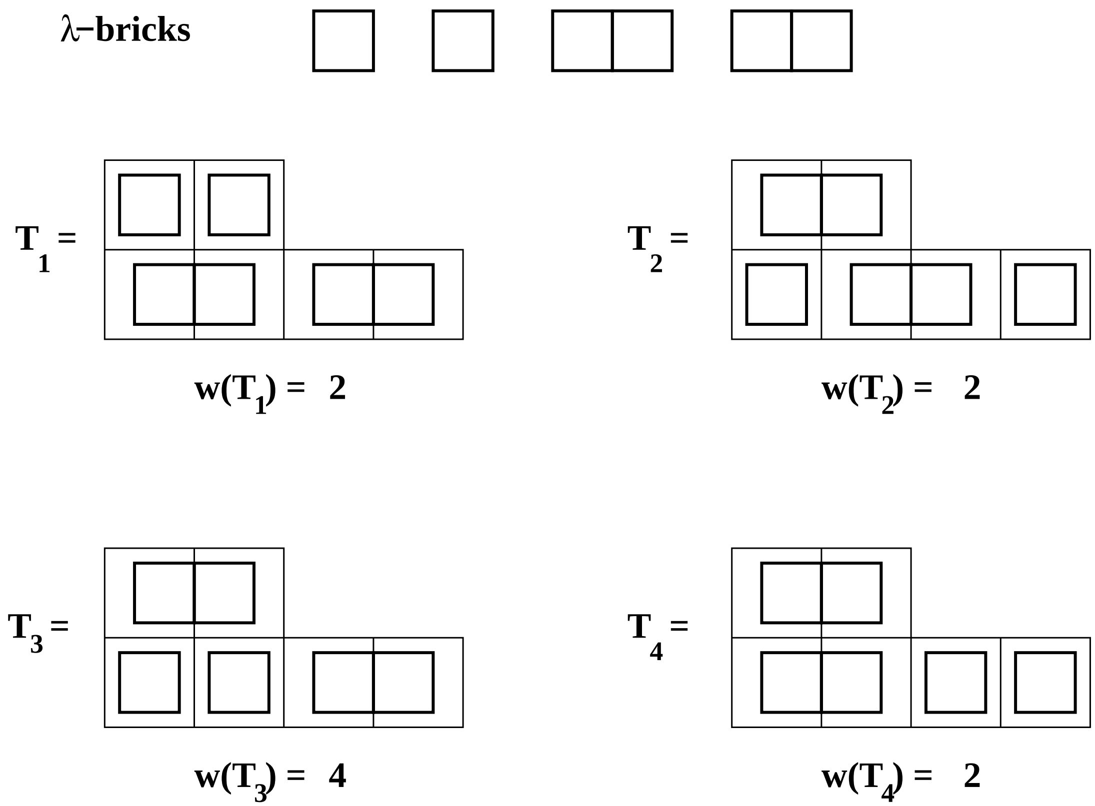
$$\begin{aligned} h_\mu(\overline{x}) &= \sum_{\mu \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,\mu} e_\lambda(\overline{x}) \\ p_\mu(\overline{x}) &= \sum_{\mu \vdash n} (-1)^{n-\ell(\lambda)} w(B)_{\lambda,\mu} e_\lambda(\overline{x}) \end{aligned}$$

(Egecioglu and Remmel 1991)

λ-Brick Tabloids and Weighted λ-Brick Tabloids.

Suppose that $\lambda = (1,1,2,2)$ and $\mu = (2,4)$.

$B_{\lambda,\mu} = 4$ and $w(B_{\lambda,\mu}) = 10$



A link from Λ to permutation enumeration

For $\sigma_1 \cdots \sigma_n \in S_n$,

$des(\sigma)$ is the number of times $\sigma_i > \sigma_{i+1}$,

$ris(\sigma)$ is the number of times $\sigma_i < \sigma_{i+1}$ where $\sigma_{n+1} = n + 1$.

Ex. Let $\sigma = 12 \;\; 9 \;\; 7 \;\; 2 \;\; 6 \;\; 8 \;\; 10 \;\; 1 \;\; 3 \;\; 4 \;\; 11 \;\; 5$.

Then $des(\sigma) = 5$ and $ris(\sigma) = 7$.

Let $f_1 : \{0, 1, \ldots\} \to \mathbb{Q}[x, y]$ such that:

$$f_1(n) = \begin{cases} 1 & \text{if } n = 0 \\ (-y)\,(x-y)^{n-1} & \text{if } n \geq 1. \end{cases}$$

Define $\xi^{f_1} : \Lambda \to \mathbb{Q}[x, y]$ as a homomorphism such that

$$\xi^{f_1}(e_n) = \frac{(-1)^n}{n!} f_1(n).$$

Theorem.

$$n!\xi^{f_1}(h_n) = \sum_{\sigma \in S_n} x^{des(\sigma)} y^{ris(\sigma)}.$$

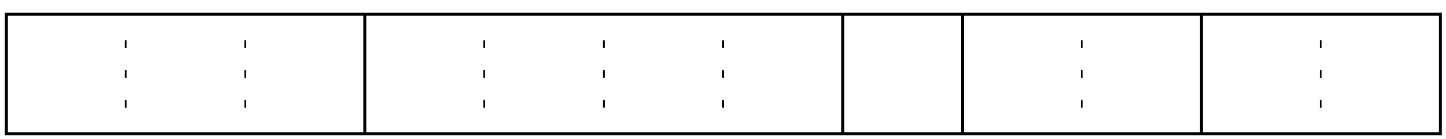
Proof.

$$
\begin{aligned}
n!\xi^{f_1}(h_n) &= n!\sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}B_{\lambda,n}\xi^{f_1}(e_\lambda)\\
&= n!\sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}B_{\lambda,n}\prod_{i=1}^{\ell(\lambda)}\frac{(-1)^{\lambda_i}}{\lambda_i!}f_1(\lambda_i)\\
&= \sum_{\lambda\vdash n}\binom{n}{\lambda_1,\ldots,\lambda_\ell}B_{\lambda,n}(-1)^{\ell(\lambda)}f_1(\lambda_1)\cdots f_1(\lambda_\ell).
\end{aligned}
$$

We have

$$n!\xi^{f_1}(h_n) = \sum_{\lambda\vdash n} \binom{n}{\lambda_1,\ldots,\lambda_\ell} B_{\lambda,n}(-1)^{\ell(\lambda)} f_1(\lambda_1)\cdots f_1(\lambda_\ell)$$

from which we create the following objects:



We have

$$n!\xi^{f_1}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell} B_{\lambda,n} (-1)^{\ell(\lambda)} f_1(\lambda_1) \cdots f_1(\lambda_\ell)$$

from which we create the following objects:

11	6	2	10	5	3	1	8	12	9	7	4

We have

$$n!\xi^{f_1}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell} B_{\lambda,n} (-1)^{\ell(\lambda)} f_1(\lambda_1) \cdots f_1(\lambda_\ell)$$

from which we create the following objects:

x	x	$-y$	$-y$	x	$-y$	$-y$	$-y$	$-y$	$-y$	x	$-y$
11	6	2	10	5	3	1	8	12	9	7	4

$f_1(n) = (-y)(x-y)^{n-1}$ for $n \geq 1$

We have

$$n!\xi^{f_1}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell} B_{\lambda,n} (-1)^{\ell(\lambda)} f_1(\lambda_1) \cdots f_1(\lambda_\ell)$$

from which we create the following objects:

x	x	y	$-y$	x	$-y$	y	y	$-y$	y	x	y
11	6	2	10	5	3	1	8	12	9	7	4

Let $\mathcal{T}_{f_1}$ be the set of objects created in this way.

The weight of $T \in \mathcal{T}_{f_1}$, $w(T)$, is the product of x,$-y$, and y labels.

$$n!\xi^{f_1}(h_n) = \sum_{T \in \mathcal{T}_{f_1}} w(T).$$

An involution will rid us of all $T \in \mathcal{T}_{f_1}$ with negative weights.

Scan left to right for a "$-y$" or two consecutive bricks with a decrease between them:

x	x	y	$-y$	x	$-y$	y	y	$-y$	y	x	y
11	6	2	10	5	3	1	8	12	9	7	4

If a $-y$ is found, break the brick in two and change $-y$ to y.

If a decrease between two bricks is found, combine the bricks and change y to $-y$. In this way,

x	x	y	$-y$	x	$-y$	y	y	$-y$	y	x	y
11	6	2	10	5	3	1	8	12	9	7	4

is sent to

x	x	y	y	x	$-y$	y	y	$-y$	y	x	y
11	6	2	10	5	3	1	8	12	9	7	4

A fixed point under this involution:

x	x	y	x	x	x	y	y	x	y	x	y
11	6	2	10	5	3	1	8	12	7	9	4

A fixed point can be read as an element in S_n:

11 6 2 10 5 3 1 8 12 7 9 4

Therefore,

$$n!\xi^{f_1}(h_n) = \sum_{T \in \mathcal{T}_{f_1}} w(T) = \sum_{\substack{T \text{ is a} \\ \text{fixed point}}} w(T) = \sum_{\sigma \in S_n} x^{des(\sigma)} y^{ris(\sigma)}.$$

□

This gives a generating function:

$$\begin{aligned}
\sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma\in S_n} x^{des(\sigma)} y^{ris(\sigma)} &= \xi^{f_1}\left(\sum_{n\geq 0} h_n t^n\right) \\
&= \xi^{f_1}\left(\sum_{n\geq 0} e_n (-t)^n\right)^{-1} \\
&= \left(\sum_{n\geq 0} \frac{t^n}{n!} f_1(n)\right)^{-1} \\
&= \frac{x-y}{x-ye^{t(x-y)}}.
\end{aligned}$$

We just

1. Defined f on $\{0, 1, \dots\}$ and ξ^f on Λ such that

$$\xi^f(e_n) = \frac{(-1)^n}{n!} f(n)$$

2. Applied ξ^f to $n!h_n$ and decorated brick tabloids

3. Performed an involution to find objects corresponding to permutations

4. Found a generating function from the h_n and e_n relationship

τ-matches

Given a sequence $\sigma = \sigma_1 \cdots \sigma_n$ of distinct integers, let $\mathrm{red}(\sigma)$ be the permutation found by replacing the i^{th} largest integer that appears in σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $\mathrm{red}(\sigma) = 1\ 4\ 3\ 2$.

Given a permutation τ in the symmetric group S_j, define a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ to have a **τ-match at place i** provided $\mathrm{red}(\sigma_i \cdots \sigma_{i+j-1}) = \tau$. Let $\tau\text{-mch}(\sigma)$ be the number of τ-matches in the permutation σ.

Let $\tau\text{-nlap}(\sigma)$ be the maximum number of nonoverlapping τ-matches in σ where two τ-matches are said to overlap if they contain any of the same integers.

Kitaev's Theorem

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\sigma)} = \frac{A(t)}{(1-x) + x(1-t)A(t)} \tag{2}$$

where $A(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}|$.

Suppose $\Upsilon \subseteq S_j$.

We say that a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has an Υ-match at place i provided $\text{red}(\sigma_i \cdots \sigma_{i+j-1}) \in \Upsilon$.

Let Υ-mch(σ) and Υ-nlap(σ) be the number of Υ-matches and nonoverlapping Υ matches in σ, respectively.

Theorem 0.1.

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n} x^{\Upsilon - nlap(\sigma)} q^{inv(\sigma)} = \frac{A_q^{\Upsilon}(t)}{(1-x) + x(1-t)A_q^{\Upsilon}(t)} \qquad (3)$$

where $A_q^{\Upsilon}(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n : \Upsilon\text{-mch}(\sigma)=0} q^{inv(\sigma)}$.

Proof of Kitaev's theorem

Suppose we fix $\tau \in S_j$.
For a permutation $\sigma \in S_n$, let

$$\mathrm{Mch}_\tau(\sigma) = \{i : \mathrm{red}(\sigma_{i+1} \cdots \sigma_{i+j}) = \tau\}$$

$I_\tau = \{1 \leq i < j :$ there exist $\sigma \in S_{j+i}$ such that $\mathrm{Mch}_\tau(\sigma) = \{0, i\}\}$.

$\alpha = 2\ 1\ 4\ 3$.

$\mathrm{Mch}_\alpha(2\ 1\ 4\ 3\ 6\ 5) = \{0, 2\}$
$\mathrm{Mch}_\alpha(3\ 2\ 5\ 4\ 1\ 7\ 6) = \{0, 3\}$. Therefore $2, 3 \in I_\alpha$.

$I_\alpha = \{2, 3\}$.

Let I_τ^* be the set of all words with letters in the set I_τ.
We let ϵ denote the empty word.

If $w = w_1 \cdots w_n \in I_\tau^*$ is word with n-letters, we define

$$\ell(w) = n, \qquad \sum w = \sum_{i=1}^{n} w_i, \qquad \text{and} \qquad ||w|| = j + \sum w.$$

In the special case where $w = \epsilon$, we let $\ell(w) = 0$ and $\sum w = 0$. Let

Let

$$A_\tau = \{w \in I_\tau^* : \ell(w) \geq 2 \text{ and } \sum w < j\} \qquad \text{and}$$

$$B_{u,\tau} = \{w_1 \cdots w_n \in I_\tau^* : \sum w_2 \cdots w_n + \sum u < j \leq \sum w_1 \cdots w_n + \sum u\}$$

for each word $u \in I_\tau^*$ with $\sum u < j$.

$\gamma = 2\ 1\ 4\ 3\ 6\ 5$

$I_\gamma = \{2, 5\}$

$A_\gamma = \{22\}$

$B_{22,\gamma} = B_{5,\gamma} = \{2, 5\}$
$B_{2,\gamma} = \{22, 52, 5\}$.

$\delta = 1\ 5\ 2\ 7\ 3\ 8\ 4\ 9\ 6.$

$I_\delta = \{2, 4, 6, 8\}.$

$$
\begin{aligned}
A_\delta &= \{22, 24, 26, 42, 44, 62, 222, 224, 242, 422, 2222\},\\
B_{2,\delta} &= \{8, 26, 46, 66, 86, 44, 64, 84, 224, 424, 624, 824,\\
&\quad 82, 62, 242, 442, 642, 842, 422, 622.822, 2222, 4222, 6222, 8222\},\\
B_{4,\delta} &= B_{22,\delta} = \{8, 6, 24, 64, 84, 82, 62, 42, 222, 422, 622, 822\},\\
B_{6,\delta} &= B_{24,\delta} = B_{42,\delta} = B_{222,\delta} = \{4, 6, 8, 22, 42, 62, 82\}, \qquad \text{and}\\
B_{8,\delta} &= B_{26,\delta} = B_{44,\delta} = B_{62,\delta} = B_{224,\delta} = B_{242,\delta} = B_{422,\delta} = \{2, 4, 6, 8\}.
\end{aligned}
$$

Form a new alphabet

$$K_\tau = \{\overline{u} : u \in I_\tau\} \cup \{\overline{w} : w \in A_\tau\}.$$

We let $\Psi : K_\tau^* \to I_\tau^*$ be the function such that $\Psi(\epsilon) = \epsilon$ and $\Psi(\overline{w_1} \cdots \overline{w_n}) = w_1 \ldots w_n$.

For example, if $\tau = \gamma = 2\ 1\ 4\ 3\ 5\ 6$ as above, then

$$\Psi(\overline{5}\ \overline{22}\ \overline{2}\ \overline{2}\ \overline{5}) = 522225.$$

Define $\overline{J}_\tau$ in the following manner.

1. $\epsilon \in \overline{J}_\tau$.
2. $\overline{v} \in \overline{J}_\tau$ for all $v \in I_\tau$.
3. If $\overline{w_1} \cdots \overline{w_n} \in \overline{J}_\tau$, then $\overline{u}\ \overline{w_1} \cdots \overline{w_n} \in \overline{J}_\tau$ for all $u \in B_{w_1,\tau}$.
4. The only words in $\overline{J}_\tau$ are the result of applying one of the above rules.

Take $J_\tau = \Psi(\overline{J}_\tau)$.

$\tau = \gamma = 2\ 1\ 4\ 3\ 5\ 6$. Then

$\overline{J}_\gamma = \{\epsilon, \overline{2}, \overline{5}, \overline{22}\ \overline{2}, \overline{52}\ \overline{2}, \overline{5}\ \overline{2}, \overline{2}\ \overline{5}, \overline{5}\ \overline{5}, \overline{2}\ \overline{22}\ \overline{2}, \overline{5}\ \overline{22}\ \overline{2}, \overline{2}\ \overline{52}\ \overline{2}, \overline{5}\ \overline{52}\ \overline{2} \ldots\}$

Let

$$\mathcal{P}_w^\tau = \{\sigma \in S_{||w||} : \mathrm{Mch}_\tau(\sigma) = \{0, w_1, w_1{+}w_2, \ldots, w_1{+}w_2{+}\cdots{+}w_n\}\}$$

and $\mathcal{P}_\epsilon^\tau = \{\tau\}$.

Key observations. Suppose we are given
$\overline{u} = \overline{u}_k \cdots \overline{u}_1 \in \overline{J}_\tau$ and $\Psi(\overline{u}) = w_1 \cdots w_n$.

1) $w = w_1 \cdots w_n \in J_\tau$.

2) If $\sigma \in \mathcal{P}_w^\tau$, then

$$\mathrm{Mch}_\tau(\sigma) = \{0, w_1, w_1 + w_2, \ldots, w_1 + w_2 + \cdots + w_n\}.$$

We can scan σ to discover that there are τ-matches at positions $1, 1 + w_1, 1 + w_1 + w_2, \ldots, 1 + w_1 + w_2 + \cdots + w_n$ so that we can recover $w_1, \ldots, w_n$ from σ.

3) We claim that $\overline{u}$ can also be recovered.

As an example, consider $w = 222252222225 \in J_\gamma$ where $\gamma = 2\ 1\ 4\ 3\ 6\ 5$. The algorithm would proceed as follows.

Step 1. $\overline{u}_1 = \overline{5}$.

Step 2. $\overline{u}_2 = \overline{2}$ since $2 + 5 \geq 6$.

Step 3. $\overline{u}_3 = \overline{22}$ since $(2 + 2) + 2 \geq 6$.

Step 4. $\overline{u}_4 = \overline{2}$ since $2 + (2 + 2) \geq 6$.

Step 5. $\overline{u}_5 = \overline{22}$ since $(2 + 2) + 2 \geq 6$.

Step 6. $\overline{u}_6 = \overline{5}$ since $5 + (2 + 2) \geq 6$.

Step 7. $\overline{u}_7 = \overline{2}$ since $2 + 5 \geq 6$.

Step 8. $\overline{u}_8 = \overline{22}$ since $(2 + 2) + 2 \geq 6$.

Step 9. $\overline{u}_9 = \overline{2}$ since $2 + (2 + 2) \geq 6$.

Thus $w = \Psi(\overline{2}\ \overline{22}\ \overline{2}\ \overline{5}\ \overline{22}\ \overline{2}\ \overline{22}\ \overline{2}\ \overline{5})$.

The Ring Homomorphism ξ

Let

$$\begin{aligned} f(n) &= (-1)^n \text{ if } n = 0, 1 \\ f(n) &= (1-x) \sum_{\omega \in J_\tau, ||\omega|| = n} (-1)^{\overline{\ell}(\omega)} \left|\mathcal{P}_\omega^\tau\right| \text{ for } n \geq 2. \end{aligned}$$

Let ξ be the ring homomorphism on Λ with the property that

$$\xi(e_n) = \frac{(-1)^n}{n!} f(n).$$

$$
\begin{aligned}
n!\xi(h_n) &= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \xi(e_\lambda) \\
&= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} \frac{(-1)^{\lambda_i}}{\lambda_i!} f(\lambda_i) \\
&= \sum_{\lambda \vdash n} \binom{n}{\lambda} (-1)^{\ell(\lambda)} B_{\lambda,n} f(\lambda_1) \cdots f(\lambda_\ell). \qquad (4)
\end{aligned}
$$

1) Use the $B_{\lambda,n}$ term to choose a brick tabloid T of shape (n) filled with bricks $b_1, \ldots, b_\ell$, reading from left to right, that induce the partition λ.

2) Use $\binom{n}{\lambda}$ to select pairwise disjoint subsets of $\{1, \ldots, n\}$ of size $|b_i|$ to assign to the bricks b_i for $i = 1, \ldots, \ell$.

3) We are left with factors of the form $(-1)^{\ell(\lambda)}$ and $f(\lambda_1) \cdots f(\lambda_\ell)$ to aid in the construction of elements in the set $\mathcal{T}_\tau$.

3a) **Bricks of length 1.**
We have -1 coming from the factor of the form $(-1)^{\ell(\lambda)}$. and a factor of $-1 = f(1)$ coming from $f(\lambda_1)\cdots f(\lambda_\ell)$.

Thus each brick of size 1 is filled with a number and has weight 1.

3b) **Bricks b_i with $|b_i| > 1$**

We use the $f(|b_i|)$ term to do the following things.

(i) pick $w \in J_\tau$ such that $||w|| = |b_i|$ and select $\sigma \in \mathcal{P}_w^\tau$.

(ii) Reorder the elements assigned to b_i so that the numbers are equal to σ when written as a permutation of $1, \ldots, |b_i|$.

(iii) If $w = i_1 i_2 \cdots i_k$, then let $\overline{u} = \overline{u}_1 \cdots \overline{u}_t$ be the word in $\overline{J}_\tau$ such that $\Psi(\overline{u}) = w$ and let $u_i = \Psi(\overline{u}_i)$ and $j_i = \sum u_i$ for $i = 1, \ldots, t$.

Place a -1 on top of the cells $j_1, j_1 + j_2, \cdots, j_1 + j_2 + \cdots j_t$ in b_i. This accounts for the $(-1)^{\overline{\ell}(w)}$ term.

(iv) Finally, the product of the -1 coming from the $(-1)^{\ell(\lambda)}$ and the term $(1 - x)$ coming from $f(|b_i|)$, leaves us with an $x - 1$ term. Thus we make the choice of either placing an x or a -1 on the last cell in the brick.

To re-cap, our construction gives a that elements $T \in \mathcal{T}_\tau$ are brick tabloids filled such that

- each integer between 1 and n appears once in T,
- a brick of length 1 contains one integer,
- a brick b of length $m \geq 2$ contains an ordered sequence of integers which reduces to σ for some $\sigma \in \mathcal{P}_w^\tau$ such that $w = i_1 i_2 \cdots i_k \in J_\tau$ and $||w|| = m$. We can then find $\overline{u} = \overline{u}_1 \cdots \overline{u}_t \in \overline{J}_\tau$ such that $\Psi(\overline{u}) = w$, and set $u_i = \Psi(\overline{u}_i)$ and $j_i = \sum u_i$ for $i = 1, \ldots, t$. Then there are -1 on top of the cells $j_1, j_1 + j_2, \ldots, j_1 + j_2 + \cdots j_t$ in b and a choice of either x or -1 for the terminal cell of b.
- if there is no $w \in J_\tau$ such that $||w|| = m$, then there are no bricks of length m.

For instance, suppose that $\tau = 1\ 3\ 2$. Then it is easy to see that $I_\tau = \{2\}$ and $J_\tau = I_\tau^* = \{2\}^*$. A brick tabloid $T \in \mathcal{T}_\tau$ may be found below.

		-1			x					-1	
7	3	11	4	10	8	6	1	5	12	9	2

Define the weight of $T \in \mathcal{T}_\tau$, $w(T)$, to be the product of all of the powers of x and -1 in the tabloid. From our construction, $n!\xi(h_n) = \sum_{T \in \mathcal{T}_\tau} w(T)$.

The sign-reversing involution $\mathfrak{I}_\tau$

Scan the bricks of $T \in \mathcal{T}_\tau$ from left to right looking for the first of the following situations:

Case 1. j consecutive bricks of length one such that the integers in these j bricks form a τ-match.

Case 2. a brick b of length j with a weight of -1

Case 3. i bricks of length 1 followed by a brick b of length $m \geq 2$ such that
(i) b contains a sequence of integers which reduces to some $\sigma \in \mathcal{P}^{\tau}_{w}$ where $w = w_1 w_2 \cdots w_k \in J_\tau$ and $||w|| = m$,
(ii) there is a word $\overline{u}_1 \cdots \overline{u}_r \in \overline{J}_\tau$ such that $\Psi(\overline{u}_1 \cdots \overline{u}_r) = w_1 w_2 \cdots w_k$ and $\Psi(\overline{u}_1) = u_1 \in A_\tau$.
(iii) if one concatenates the integers in the i bricks before b with the ordered sequence in b, this sequence reduces to some $\alpha \in P_v$ where $v = v_1 \cdots v_i w_1 \cdots w_k \in J_\tau$ and there is a word $\overline{u}\,\overline{u}_1 \cdots \overline{u}_r \in \overline{J}_\tau$ such that $\Psi(\overline{u}\,\overline{u}_1 \cdots \overline{u}_r) = v_1 \cdots v_i w_1 \cdots w_k$ and $\Psi(\overline{u}) = v_1 \cdots v_i \in B_{\tau,u_1}$.

Case 4. a brick b' of length $m > j$.

		-1			x					-1	
7	3	11	4	10	8	6	1	5	12	9	2

					x					-1	
7	3	11	4	10	8	6	1	5	12	9	2

Fixed points of $\mathfrak{I}_\tau$

Tabloids which have only

(i) bricks of length one or

(ii) bricks of length j with a weight of x whose integers form a τ-match.

Each such fixed point T may be associated with a permutation σ of n written in one line notation by reading the integers from left to right.

The number of bricks of length j weighted by x is τ-$nlap(\sigma)$.

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-}nlap(\sigma)}$$

$$= \frac{1}{1 + \sum_{n=1}^{\infty} \xi(e_n)}$$

$$= \frac{1}{1 - t + \sum_{n=2}^{\infty} \frac{t^n}{n!} (1-x) \sum_{w \in J_\tau, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\tau|}.$$

Then setting $x = 0$ in the above equations,

$$\begin{aligned} A(t) &= 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}| \\ &= \frac{1}{1 - t + \sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w \in J_\tau, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\tau|}. \end{aligned}$$

Thus

$$\sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w \in J_\tau, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\tau| = \frac{1}{A(t)} - (1-t).$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\sigma)} &= \frac{1}{1 + \sum_{n=1}^{\infty} \xi(e_n)} \\
&= \frac{1}{1 - t + (1-x)(\frac{1}{A(t)} - (1-t))} \\
&= \frac{A(t)}{(1-x) - x(1-t)A(t)}.
\end{aligned}$$

Corollary 0.2. *For any permutation* τ,

$$A(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}|$$
$$= \frac{1}{1 - t + \sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w \in J_\tau, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\tau|}.$$

Corollary 0.3. *For any permutation* τ,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\sigma)} =$$
$$\frac{1}{1 - t + (1-x)\sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w \in J_\tau, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\tau|}.$$

Example $\tau = j \cdots 2\ 1$

$I_\tau = \{1\}$, $B_{1,\tau} = \{1^{j-1}\}$.

$\overline{J}_\tau = \{\overline{1},\ \overline{1^{j-1}}\,\overline{1},\ \overline{1}\,\overline{1^{j-1}}\,\overline{1},\ \overline{1^{j-1}}\,\overline{1}\,\overline{1^{j-1}}\,\overline{1}, \ldots\}$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\sigma)} = \frac{1}{x(1-t) + (1-x)\left(\sum_{n=0}^{\infty} \frac{t^{jn}}{(jn)!} - \sum_{n=0}^{\infty} \frac{t^{jn+1}}{(jn+1)!}\right)}$$

Special cases of the above expression sometimes simplify nicely; for example, when $j = 2$:

$$\frac{e^t}{(1-x) + x(1-t)e^t},$$

and when $j = 3$:

$$\frac{e^{t/2}}{x(1-t)e^{t/2} + (1-x)\left(\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)}.$$

Example $\tau = 1\ 3\ 2$.

$I_\tau = 2$. $J_\tau = I_\tau^* = \{2\}^*$. Suppose that $\sigma_1 \ldots \sigma_{2n+3} \in P_{2^n}^\tau$.

(I) Since there is a τ-match starting at position $2i+1$ for $i = 0, \ldots, n$, it must be the case that $\sigma_{2i+1} < \sigma_{2i+2}, \sigma_{2i+3}$ for $i = 0, \ldots, n$. It follows that $\sigma_1 = 1$ and $\sigma_3 = 2$.

(II) σ_2 can be any element of $\{3, \ldots, 2n+3\}$ and that $\text{red}(\sigma_3 \ldots \sigma_{2n+3}) \in P_{2^{n-1}}^\tau$ if $n \geq 1$.

(III) $|P_{2^n}^\tau| = (2n+1)|P_{2^{n-1}}^\tau|$ if $n \geq 1$.

(IV) Since $P_{2^0}^\tau = 1$, it follows by induction that $|P_{2^n}^\tau| = (2n+1)!! = (2n+1)(2n-1)\cdots 3 \cdot 1$ for $n \geq 0$.

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \mathrm{Mch}_{132}(\sigma) = \emptyset\}| =$$

$$\frac{1}{1 - t + \sum_{n=0}^{\infty} (-1)^n |P_{2^n}^{132}| \frac{t^{2n+3}}{(2n+3)!}} =$$

$$\frac{1}{1 - t + \sum_{n=0}^{\infty} (-1)^n (\prod_{i=0}^{n} (2i+1)) \frac{t^{2n+3}}{(2n+3)!}} =$$

$$\frac{1}{1 - \int exp(-t^2/2) dt}.$$

By Kitaev's Theorem

$$1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-}nlap(\sigma)} = \left(1 - tx + (x-1) \int e^{-t^2/2}\, dt\right)^{-1}$$

in the case $\tau = 1\ 3\ 2$.

Υ-matches: $\Upsilon \subseteq S_j$

$\mathrm{Mch}_{\Upsilon}(\sigma) = \{i : \mathrm{red}(\sigma_{i+1} \cdots \sigma_{i+j}) \in \Upsilon\}$ and

$$I_{\Upsilon} = \{1 \leq i < j : \text{there exist } \sigma \in S_{j+i} \text{ such that } \mathrm{Mch}_{\Upsilon}(\sigma) = \{0, i\}\}.$$

$$A_\Upsilon = \{w \in I_\Upsilon^* : \ell(w) \geq 2 \text{ and } \sum w < j\} \qquad \text{and}$$

$$B_{u,\Upsilon} = \{w_1 \cdots w_n \in I_\Upsilon^* : \sum w_2 \cdots w_n + \sum u < j \leq \sum w_1 \cdots w_n + \sum u\}$$

for each word $u \in I_\Upsilon^*$ with $\sum u < j$.

We also set

$$\mathcal{P}_w^\Upsilon = \{\sigma \in S_{||w||} : \mathrm{Mch}_\Upsilon(\sigma) = \{0, w_1, w_1+w_2, \ldots, w_1+w_2+\cdots+w_n\}\}.$$

Form a new alphabet

$$K_\Upsilon = \{\overline{u} : u \in I_\Upsilon\} \cup \{\overline{w} : w \in A_\Upsilon\}.$$

We define the natural map $\Psi : K_\Upsilon^* \to I_\Upsilon^*$ such that $\Psi(\epsilon) = \epsilon$ and $\Psi(\overline{w_1} \cdots \overline{w_n}) = w_1 \ldots w_n$. Define $\overline{J}_\Upsilon$ recursively as follows.

1. $\epsilon \in \overline{J}_\Upsilon$.
2. $\overline{v} \in \overline{J}_\Upsilon$ for all $v \in I_\Upsilon$.
3. If $\overline{w_1} \cdots \overline{w_n} \in \overline{J}_\Upsilon$, then $\overline{u}\ \overline{w_1} \cdots \overline{w_n} \in \overline{J}_\Upsilon$ for all $u \in B_{w_1,\Upsilon}$.
4. The only words in $\overline{J}_\Upsilon$ are the result of applying one of the above rules.

Let $J_\Upsilon = \Psi(\overline{J}_\Upsilon)$.

The Homomorphism $\xi_q(n)$

$$\begin{aligned} f_q(n) &= (-1)^n \text{ if } n = 0, 1 \\ f_q(n) &= (1-x) \sum_{\|\omega\|=n, \omega \in J_\Upsilon} (-1)^{\overline{\ell}(\omega)} \sum_{\sigma \in \mathcal{P}_\omega^\Upsilon} q^{inv(\sigma)}, n \geq 2. \end{aligned}$$

otherwise.

$$\xi_q(e_n) = \frac{(-1)^n}{[n]_q!} f_q(n).$$

Theorem 0.4. *For any set of permutations $\Upsilon \subseteq S_j$ where $j > 1$,*

$$A_q^{\Upsilon}(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n : Mch_{\Upsilon}(\sigma)=\emptyset} q^{inv(\sigma)}$$

$$= \frac{1}{1 - t + \sum_{n=2}^{\infty} \frac{t^n}{[n]_q!} \sum_{w \in J_{\Upsilon}, ||w||=n} (-1)^{\overline{\ell}(w)} \sum_{\sigma \in \mathcal{P}_w^{\Upsilon}} q^{inv(\sigma)}}$$

and

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]!} \sum_{\sigma \in S_n} x^{\Upsilon - nlap(\sigma)} q^{inv(\sigma)} =$$

$$\frac{1}{1 - t + (1-x) \sum_{n=2}^{\infty} \frac{t^n}{[n]_q!} \sum_{w \in J_{\Upsilon}, ||w||=n} (-1)^{\overline{\ell}(w)} \sum_{\sigma \in \mathcal{P}_w^{\Upsilon} |} q^{inv(\sigma)}} =$$

$$\frac{A_q^{\Upsilon}(t)}{(1-x) + x(1-t) A_q^{\Upsilon}(t)}.$$

Υ-matchings for m-tuples of permutations.

For example, if $\Upsilon = \{3\ 1\ 2, 2\ 1\ 3\}$ and $m = 2$, then the pair

$\sigma^1 = 2\,1\,4\,3\,7\,6\,9\,8\,5\,(11)\,(10)$ and $\sigma^2 = 2\,1\,5\,3\,9\,6\,7\,8\,4\,(10)\,(11)$

has $\mathrm{Mch}^2_\Upsilon(\sigma_1, \sigma_2) = \{0, 2, 7\}$, so that (σ^1, σ^2) is an element of $\mathcal{P}^{\Upsilon,2}_{2\ \ 5}$.

The Homomorphism for m-tuples of permutations. Let $\vec{q} = (q_1, \ldots, q_m)$.

$U_{\vec{q},m}(n) = (-1)^n$ if $n = 0, 1$ and

$$U_{\vec{q},m}(n) = (1-x) \sum_{\omega \in J_{\Upsilon}^{m}, ||\omega||=n} (-1)^{\overline{\ell}(\omega)} \sum_{(\sigma^1,\ldots,\sigma^m) \in \mathcal{P}_{\omega}^{\Upsilon,m}} \prod_{i=1}^{m} q_i^{inv(\sigma^i)} \tag{5}$$

otherwise. Let $\xi_{\Upsilon,\vec{q},m}$ be the ring homomorphism on Λ with the property that

$$\xi_{\Upsilon,\vec{q},m}(e_n) = \frac{(-1)^n}{\prod_{i=1}^{m} [n]_{q_i}!} U_{\vec{q},m}(n). \tag{6}$$

Theorem 0.5. *For any set of permutations $\Upsilon \subseteq S_j$ where $j > 1$,*

$$A^{\Upsilon,\vec{q},m}(t) =$$

$$\sum_{n=0}^{\infty} \frac{t^n}{\prod_{i=0}^{m}[n]_{q_i}!} \sum_{(\sigma^1,\ldots,\sigma^m)\in S_n^m:\Upsilon\text{-}commch(\sigma^1,\ldots,\sigma^m)=0} \prod_{i=1}^{n} q_i^{inv(\sigma^i)} =$$

$$\frac{1}{1-t+\sum_{n=2}^{\infty}\frac{t^n}{\prod_{i=1}^{m}[n]_{q_i}!}\sum_{w\in J_\Upsilon,||w||=n}(-1)^{\overline{\ell}(w)}\sum_{(\sigma^1,\ldots,\sigma^m)\in\mathcal{P}_w^{\Upsilon,m}}\prod_{i=1}^{m} q_i^{inv(\sigma^i)}},$$

and

$$\sum_{n=0}^{\infty} \frac{t^n}{\prod_{i=1}^{m}[n]_{q_i}!} \sum_{(\sigma_1,\ldots,\sigma^m)\in S_n^m} x^{\Upsilon\text{-}comnlap(\sigma^1,\ldots,\sigma^m)} \prod_{i=1}^{m} q_i^{inv(\sigma^i)} =$$

$$\frac{A^{\Upsilon,\vec{q},m}(t)}{(1-x)+x(1-t)A^{\Upsilon,\vec{q},m}(t)} =$$

$$\frac{1}{1-t+(1-x)\sum_{n=2}^{\infty}\frac{t^n}{\prod_{i=1}^{m}[n]_{q_i}!}\sum_{\substack{w\in J_\Upsilon\\ ||w||=n}}(-1)^{\overline{\ell}(w)}\sum_{(\sigma^1,\ldots,\sigma^m)\in\mathcal{P}_w^{\Upsilon,m}}\prod_{i=1}^{m} q_i^{inv(\sigma^i)}}$$

Matching in Words For $m \geq 1$, let $\{1, \ldots, m\}_n^*$ be the set of all words of length n in the letters $\{1, \ldots, m\}$.

For $v \in \{1, \ldots, m\}_j^*$, we define a word $w \in \{1, \ldots, m\}_j^*$ to have a **v-match at place** i provided $w_i \cdots w_{i+j-1}$ is equal to v.

Similarly if Δ is a set of words in $\{1, \ldots, m\}_j^*$, we define a word $w \in \{1, \ldots, m\}_j^*$ to have a Δ-match at place i provided $w_i \cdots w_{i+j-1} \in \Delta$.

If $v = v_1 \cdots v_n \in \{1, \ldots, m\}^*$, we define

$\mathrm{Mch}_\Delta(v) = \{i : v_{i+1} \cdots v_{i+j} \in \Delta\}$ and

$$I_\Delta = \{1 \le i < j : \text{there exists a word } v \text{ such that } \mathrm{Mch}_\Delta(v) = \{0, i\}\}.$$

Theorem 0.6. *For any set of words* $\Delta \subseteq \{1, \ldots, m\}_j^*$ *where* $j > 1$,

$$A^{\Delta}(t) = 1 + \sum_{n=1}^{\infty} t^n |\{v \in \{1, \ldots, m\}_n^* : Mch_{\Delta}(v) = \emptyset\}|$$
$$= \frac{1}{1 - mt + \sum_{n=2}^{\infty} t^n \sum_{w \in J_{\Delta}, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^{\Delta}|},$$

and

$$\sum_{n=0}^{\infty} t^n \sum_{v \in \{1,\ldots,m\}_n^*} x^{\Delta\text{-}nlap(v)} =$$
$$\frac{1}{1 - mt + (1-x) \sum_{n=2}^{\infty} t^n \sum_{w \in J_{\Delta}, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^{\Delta}|} =$$
$$\frac{A^{\Delta}(t)}{(1-x) + x(1-mt)A^{\Delta}(t)}.$$

We note that in the special case where Δ consists of a single word v, Kitaev and Mansour proved

$$\sum_{n=0}^{\infty} t^n \sum_{v \in \{1,\ldots,m\}_n^*} x^{v\text{-}nlap(v)} = \frac{A^{\Delta}(t)}{(1-x)+x(1-mt)A^{\Delta}(t)}.$$

Example $m = 2$, $k = 2$, $\Delta = \{v\}$ where $v = 2\ 1\ 2$.
$I^2_\Delta = \{2\}$, $J^2_\Delta = \{2\}^*$, and $|\mathcal{P}^{\Delta,2}_{2^n}| = 1$ for all n.

$$A^{\Delta,2}(t) = \sum_{n=0}^{\infty} t^n |\{(u^1,u^2) \in (\{1,2\}^*)^2 : \Delta\text{-}commch(u^1,u^2) = 0\}|$$

$$= \frac{1}{1 - 4t + \sum_{n=0}^{\infty} t^{3+2n}(-1)^n}$$

$$= \frac{1+t^2}{1-4t+t^2-3t^3}.$$

Hence

$$\sum_{n=0}^{\infty} t^n \sum_{(u^1,u^2)\in(\{1,2\}^*)^2} x^{\Delta\text{-}comnlap(u^1,u^2)} = \frac{1+t^2}{1-4t+t^2-3t^3-xt^3}.$$

Finding generating gunctions for the total number of τ-matches

Suppose that $I_\tau = \{k_\tau\}$ where $2k_\tau \geq j$.

We break $A(t)$ into k_τ pieces. For $0 \leq m < k_\tau$, let
$A_m(t) = \sum_{n=1}^{\infty} t^{nk_\tau - m} \left(A(t)|_{t^{nk_\tau - m}} \right)$

$$A(t) = 1 + \sum_{m=0}^{k_\tau - 1} A_m(t)$$

$$A_m(t) = \frac{1}{k_\tau} \sum_{\ell=0}^{k_\tau - 1} e^{\frac{2\pi i}{k_\tau}\ell m} A\left(e^{\frac{2\pi i}{k_\tau}\ell} t\right)$$

For $m = 0$,
$A_0(t) = -1 + \frac{1}{k_\tau} \sum_{\ell=0}^{k_\tau - 1} A\left(e^{\frac{2\pi i}{k_\tau}} t\right)$.

Let ϑ be the homomorphism defined on e_n such that

$$\vartheta(e_n) = \begin{cases} 1 & \text{if } n = 0, \text{ and} \\ (-1)^n(-x)(1-x)^{n-1} \, A(t)|_{t^{nk_\tau}} & \text{if } n \geq 1. \end{cases}$$

Let $0 \leq m < k_\tau$. We will apply ϑ on the recursively defined symmetric function with parameter a function. This function will weight the last brick in a brick tabloid differently than the other bricks, is υ defined by

$$\upsilon(n) = \frac{A(t)|_{t^{nk_\tau - m}}}{A(t)|_{t^{nk_\tau}}}.$$

The homomorphism ϑ and the function υ can be used to prove

Theorem 0.7. *For $\tau \in S_j$, if $I_\tau = \{k_\tau\}$ where $2k_\tau \geq j$, then*

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} = \frac{\sum_{m=0}^{k_\tau - 1} (1-x)^{m/k_\tau} A_m \left(t \sqrt[k_\tau]{1-x}\right)}{(1-x) - x A_0 \left(t \sqrt[k_\tau]{1-x}\right)}.$$

Example $\tau = 1\ 3\ 2$, $I_\tau = \{2\}$.

$$A(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}| = \left(1 - \int e^{-t^2/2}\, dt\right)^{-1}.$$

$$A_0(t) = \tfrac{A(t)+A(-t)}{2} - 1$$
$$A_1(t) = \tfrac{A(t)-A(-t)}{2}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} = \left(1 - \int e^{t^2(x-1)/2}\, dt\right)^{-1}.$$