Slides for a lecture in
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Dashed Pattern Avoidance on Other Groups

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Motto

There are two ways of regarding
a permutation :

"passive" – as a linear order

"active" – as a bijection, group element

(Peter Cameron, Combinatorics)

Starting point of our tour:

University of Reykjavik

(results of the Icelandic school)

Destination:

See the melting ice

(i.e., transform the "passive" combinatorial
concept of pattern avoidance into
an "active" algebraic concept)

Dashed Patterns

Definition. `[ Babson-Steingrímsson 2000 ]`
A permutation $\pi \in S_n$ contains the pattern $(1-32)$ if $\pi = [\dots, a, \dots, c, b, \dots]$ for some $a < b < c$;

π **avoids** $(1-32)$ if no such a, b, c exist.

Let

$$S_n(1-32) := \{\pi \in S_n \mid \pi \text{ avoids } (1-32)\}.$$

Example.

$$[4,5,2,1,3] \in S_n(1-32)$$

$$[1,3,4,5,2] \notin S_n(1-32).$$

Two Theorems

Let

$$Des(\pi) := \{i \mid \pi(i) > \pi(i+1)\}$$

and

$$des(\pi) := \# Des(\pi).$$

Theorem `[Claesson '01]`

$$\sum_{\pi \in S_n(1-32)} q^{1+des(\pi)} = \sum_{k \geq 0} S(n,k) q^k,$$

where $S(n,k)$:= Stirling numbers of 2^{nd} kind.

Let

$$inv(\pi) := \#\{i < j \mid \pi(i) > \pi(j)\}$$

and

$$rmaj(\pi) := \sum_{ı \in Des(\sigma)} (n - i).$$

Theorem [**?** , Regev-R '04]

$$\sum_{\pi \in S_n(1-32)} q^{inv(\pi)} = \sum_{\pi \in S_n(1-32)} q^{rmaj(\pi^{-1})}.$$

1^{st} **$1,000 Question:**
Who was the first to prove this result ?

Signed Permutations

A *signed permutation* on n letters is a bijection

$$\pi : [-n, n] \setminus \{0\} \mapsto [-n, n] \setminus \{0\},$$

such that for every $i \in [-n, n] \setminus \{0\}$

$$\pi(-i) = -\pi(i).$$

Example. By

$$\pi = [\bar{2}, 1, \bar{3}]$$

we mean $\pi(1) = -2,\ \pi(2) = 1,\ \pi(3) = -3,$ thus $\pi(-1) = 2,\ \pi(-2) = -1,\ \pi(-3) = 3.$

B_n – the group of signed permutations on n letters

Coxeter Groups

S_n – the symmetric group on n letters

is generated by the adjacent transpositions

$s_1, \ldots, s_{n-1}$, where $s_i := (i, i+1)$,

with the defining relations

$$(s_i s_j)^2 = 1 \qquad (|i-j| > 1);$$

$$(s_i s_{i+1})^3 = 1 \qquad (1 \le i < n-2);$$

$$s_i^2 = 1 \qquad (1 \le i \le n-1).$$

B_n – the group of signed permutations on n letters is generated by $s_0, s_1, \ldots, s_{n-1}$ with the above defining relations and

$$(s_0 s_j)^2 = 1 \qquad (j > 1); \qquad (s_0 s_1)^4 = 1;$$

$$s_0^2 = 1.$$

Statistics on Coxeter Groups

For $\pi \in S_n$ let

$\ell_S(\pi) :=$ the length of π w.r.t $s_1, \ldots, s_{n-1}$.

Then

$$inv(\pi) := \#\{i > j \mid \pi(i) > \pi(j)\} = \ell_S(\pi),$$

and

$$Des_S(\pi) := \{i \mid \pi(i) > \pi(i+1)\} =$$

$$= \{i \mid \ell_S(\pi s_i) \le \ell_S(\pi)\}.$$

Hence

$$des(\pi) = \#\{i \mid \ell_S(\pi s_i) \le \ell_S(\pi)\}.$$

Length and descent number on a general Coxeter group are defined similarly.

Q1: How to define maj on a general Coxeter group ?

Q12: How to define 1−32 avoidance on other Coxeter groups ?

Statistics on signed permutations

Definition. A permutation $\sigma \in B_n$ contains the pattern $(2, |1|)$ if $\sigma = [\ldots, b, a, \ldots]$ or $\sigma = [\ldots, b, \bar{a}, \ldots]$ for some $a < b$;
$\sigma \in B_n$ contains $(|1|, \bar{3} - |2|)$ if

$$\sigma = [\ldots, a, \bar{c}, \ldots, b, \ldots] \text{ or } [\ldots, \bar{a}, \bar{c}, \ldots, b, \ldots]$$

$$\text{or } [\ldots, a, \bar{c}, \ldots, \bar{b}, \ldots] \text{ or } [\ldots, \bar{a}, \bar{c}, \ldots, \bar{b}, \ldots].$$

Definition. `flag-major index [Adin-R '99]`

For $\pi \in B_n$ let

$$fmaj(\pi) := 2 \cdot maj(\pi) + \#\{i \mid \pi(i) < 0\}.$$

Example. $maj[4, \bar{3}, \bar{2}, \bar{1}] = 1$, since $4 > -3 < -2 < -1$. Thus $fmaj[4, \bar{3}, \bar{2}, \bar{1}] = 2 \cdot 1 + 3$

Two Theorems on B_n

Theorem [Regev-R '05]

$$\sum_{\pi \in B_n(2|1|,\ |1|\bar{3}-|2|,\ |2|\bar{3}-|1|)} q^{1+des(\pi)} =$$

$$\sum_{k \geq 0} S_B(n,k) q^k,$$

where $S_B(n,k) :=$ type B Stirling numbers of 2^{nd} kind.

Theorem [? , Regev-R '06]

$$\sum_{\pi \in B_n(\pi \in B_n(2|1|,\ |1|\bar{3}-|2|,\ |2|\bar{3}-|1|)} q^{\ell_B(\pi)} =$$

$$\sum_{\pi \in B_n(\pi \in B_n(2|1|,\ |1|\bar{3}-|2|,\ |2|\bar{3}-|1|)} q^{fmaj(\pi^{-1})},$$

where $\ell_B(\pi) :=$ the length w.r.t. the Coxeter generators.

2^{nd} **\$1,000 Question:**
Who was the first to prove this result ?

The Alternating Group A_n is nearly Coxeter

Let $a_i := s_1 s_{i+1} = (1,2)(i,i+1) \in A_n$.

Theorem `[Bourbaki, Mitsuhashi '01]`
The set

$$A := \{a_i \mid 1 \le i \le n-2\}$$

generates A_n with the defining relations

$$(a_i a_j)^2 = 1 \qquad (|i-j| > 1);$$

$$(a_i a_{i+1})^3 = 1 \qquad (1 \le i < n-2);$$

$a_1^3 = 1$ and $a_i^2 = 1$ $(1 < i \le n-2)$.

Equidistribution on A_n

Definitions. For $\pi \in A_n$ let

$\ell_A(\pi) :=$ the length of π w.r.t $A \cup A^{-1}$.

$$Des_A(\pi) := \{i \mid \ell_A(\pi a_i) \le \ell_A(\pi)\}$$

$$rmaj_A(\pi) \sum_{i \mid i \in Des_A(\pi)} (n-1-i).$$

Theorem [**?** , Regev-R '04]

$$\sum_{\pi \in A_n(1-2-43,\ 2-1-43)} q^{\ell_A(\pi)} =$$

$$\sum_{\pi \in A_n(1-2-43,\ 2-1-43)} q^{rmaj_A(\pi^{-1})}.$$

3rd **\$1,000 Question:**
Who was the first to prove this result ?

Björner-Wachs Theorem

Let

$$lr(\pi) := \# \text{ left to right minima of } \pi.$$

Theorem `[BW '91]`

$$\sum_{\pi \in S_n} q_1^{inv(\pi)} q_2^{lr(\pi)} q_3^{des(\pi)} =$$

$$\sum_{\pi \in S_n} q_1^{rmaj(\pi^{-1})} q_2^{lr(\pi)} q_3^{des(\pi)}.$$

$$B_n$$

Theorem `[Adin-Brenti-R '04]`

$$\sum_{\pi \in B_n} q^{\ell_B(\pi)} t^{des(\pi)} =$$

$$\sum_{\pi \in B_n} q^{fmaj(\pi^{-1})} t^{des(\pi)}.$$

For $\pi \in B_n$ let

$$nrl(\pi) :=$$

$$\#\{1 \leq i \leq n \mid \forall j > i \; |\pi(j)| > |\pi(i)|, \; \pi(i) < 0\}.$$

Example. $nrl[\bar{4}, \bar{5}, 2, 3, \bar{1}] = 2$, since $5, 3, 1$ are right to left maxima, but 3 is not negative.

Theorem `[Regev-R '05]`

$$\sum_{\pi \in B_n} q^{\ell_B(\pi)} t^{nrl(\pi)} =$$

$$\sum_{\pi \in B_n} q^{fmaj(\pi^{-1})} t^{nrl(\pi)}.$$

Theorem `[Foata-Han '05]`

$$\sum_{\pi \in B_n} q_1^{\ell_B(\pi)} q_2^{nrl(\pi)} q_3^{des(\pi)} =$$

$$\sum_{\pi \in B_n} q_1^{fmaj(\pi^{-1})} q_2^{nrl(\pi)} q_3^{des(\pi)}.$$

BW for A_n

A position i is an `almost left to right minimum` of $\pi \in S_n$

if there is **at most one** $j < i$,

such that $\pi(j) < \pi(i)$,

For $\pi \in A_n$ let

$alr(\pi) :=$ # almost l.t.r. minima.

Example. $alr[4, 2, 3, 5, 1] = 4$.

Theorem `[Regev-R '04]`

$$\sum_{\pi \in A_n} q_1^{\ell_A(\pi)} q_2^{alr(\pi)} q_3^{des_A(\pi)} =$$

$$\sum_{\pi \in A_n} q_1^{rmaj_A(\pi^{-1})} q_2^{alr(\pi)} q_3^{des_A(\pi)}.$$

Observations [Regev-R]

$$S_n(1-32) =$$

$$\{\pi \in S_n \mid 1 + des(\pi) = lr(\pi)\ \}.$$

$$A_n(1-2-43,\ 2-1-43) =$$

$$\{\pi \in A_n \mid 2 + des_A(\pi) = alr(\pi)\ \}.$$

$$B_n(2|1|,\ |1|\bar{3}-|2|,\ |2|\bar{3}-|1|) =$$

$$\{\pi \in B_n \mid des_A(\pi) = nrl(\pi)\ \}$$

Answers to the $1,000 Questions

First, implicit in Björner-Wachs

Second, implicit in Foata-Han

Third, explicit in Regev-R

Extending the Concepts - I

Given a flag

$$G_1 < G_2 < G_3 < G_4 < \cdots$$

Let R_i be the set of shortest coset representatives of G_i in G_{i+1}.

Let R_i^{max} be the subset of longest elements in R_i.

The `delent number` of $\pi \in G_n$ is the number of factors from R_i^{max} needed to express π.

Theorem `[Regev-R]`

$$delent(\pi) = \begin{cases} lr(\pi) - 1, & \pi \in S_n \ ; \\ nrl(\pi), & \pi \in B_n \ ; \\ alr(\pi) - 2, & \pi \in A_n \ . \end{cases}$$

Conclusion: $lr, \ alr, nrl$ are different occurrences of a **unified concept**.

Extending the Concepts - II

(the flag major index)

Similarly,

$$maj,\ fmaj, rmaj_A$$

are different occurrences
of a **unified concept**

Theorem `[R-Shwartz '06]` Every finite Weyl group W (E_8...not yet)
with degrees $d_1, \ldots, d_t$ is a product of t cyclic subgroups

$$W = C_1 \cdots C_t,$$

where $C_i \cong \boldsymbol{Z}_{d_i}$.

Thus every element $w \in W$ has a unique presentation

$$w = \prod v_i^{k_i}$$

where $0 \leq k_i \leq d_i - 1$.

The sum of the exponents gives the various major indices.

Other Groups

Wreath Products – Adin-R, Regev-R, Remmel-R

D_n – Biagioli-Caselli

Complex reflection groups – Bagno-Biagioli

Alternating subgroups of other Coxeter groups – Brenti-Reiner-R

Future

Other patterns

Other groups

Representation theory

... ???