Layered permutations and rational generating functions

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Outline

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Moral:

It can be better to count by containment instead of avoidance.

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Note

$$w \text{ is a composition iff } w \in \mathbb{P}^*.$$

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Given $u \leq w$ there is a unique *rightmost embedding, I,* such that $I \geq I'$ componentwise for all embeddings I'. The embedding above is rightmost.

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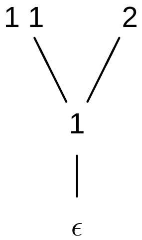

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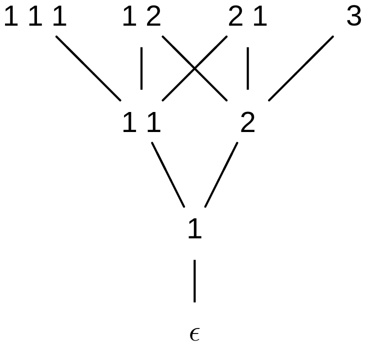

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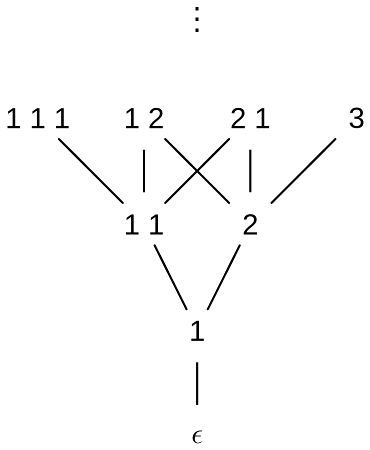

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For any alphabet A, the *formal power series in noncommuting variables A with integral coefficients* is

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Convention: If $S \subseteq A$, then we also let S stand for $\sum_{s \in S} s$.

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Now if $u = \bar{k}_1 \ldots \bar{k}_r$ then

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If $u \in [\bar{n}]^*$ has $t(u) = (k_1, \ldots, k_n)$ then

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If π and π' are layered permutations with the same multiset of layer lengths then for all $n \geq 0$:

$$\#\mathfrak{S}_n(231, 312, \pi) = \#\mathfrak{S}_n(231, 312, \pi'). \qquad \blacksquare$$

Outline

Compositions and layered permutations

Rational generating functions

Commuting variables and a Wilf equivalence

Comments and open questions

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In fact, they give a construction to compute the generating function. Can this method be used to prove the Wilf equivalence? See also the work of Mansour and Egge.

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4. One can also consider the Möbius function of P^* (Vatter and Sagan) and various interesting subposets of P^* (Goyt).

ÞAKKA YKKUR KÆRLEGA FYRIR!