Separable d-permutations and Guillotine Partitions

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Overview

- Floorplane partitions and guillotine partitions

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- Separable d-permutations

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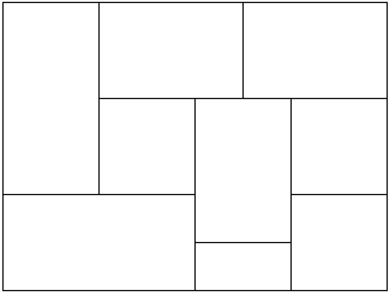
- Floorplane partitions and guillotine partitions
- Separable d-permutations
- Restricted guillotine partitions

Floorplan partitions

A floorplan partition: A partition of a rectangle into smaller rectangles.



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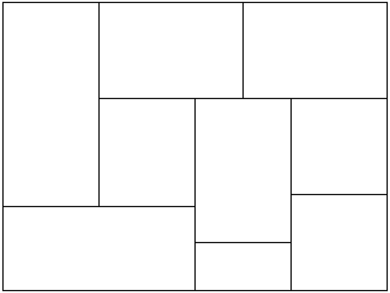

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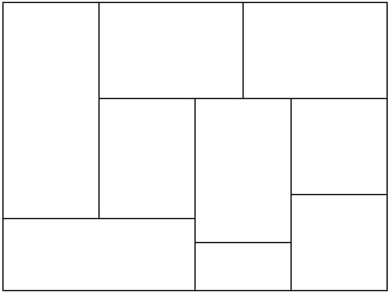

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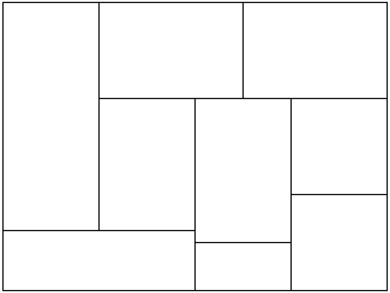

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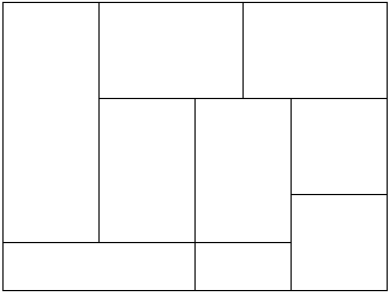

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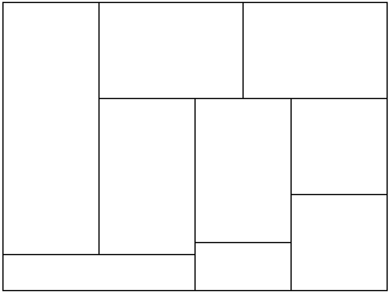

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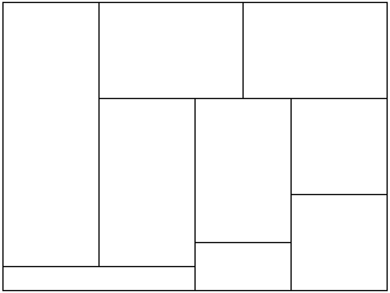

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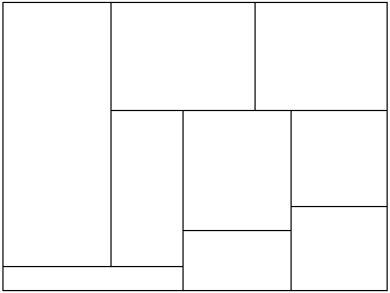

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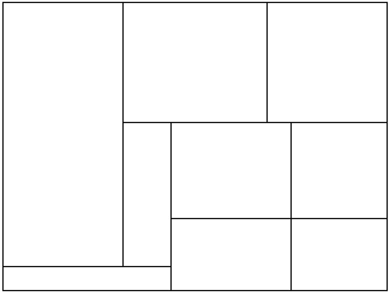

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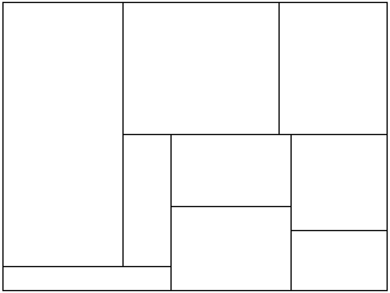

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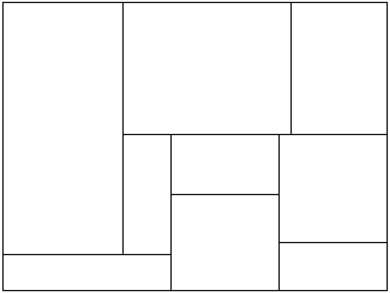

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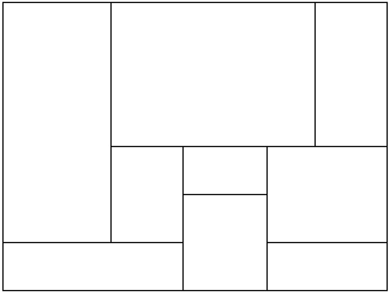

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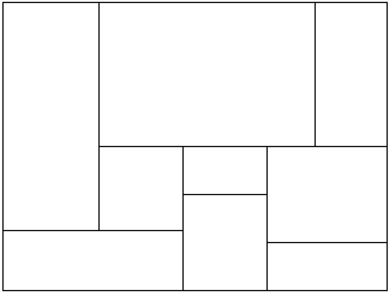

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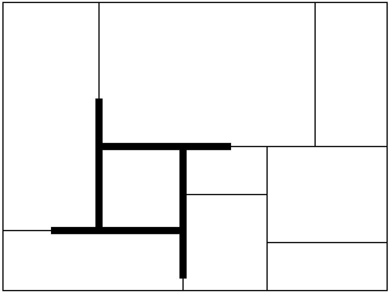
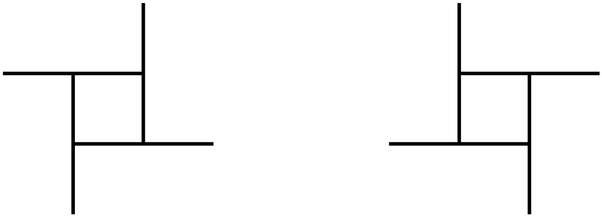
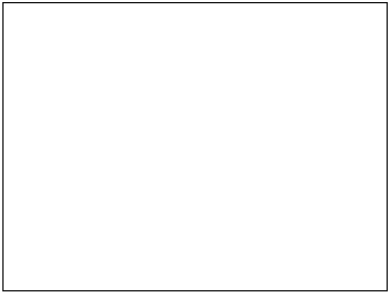


Pin wheel structure



Guillotine partitions

Either the empty partition...



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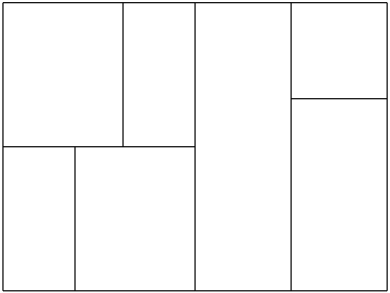

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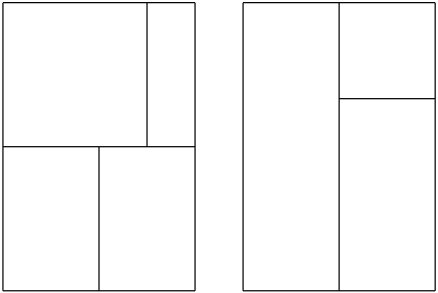

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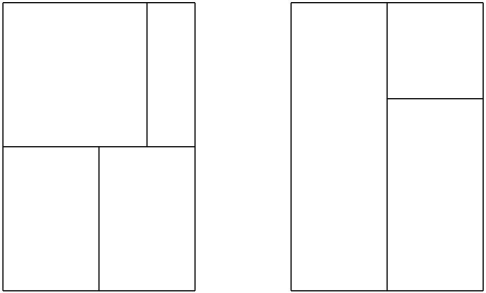

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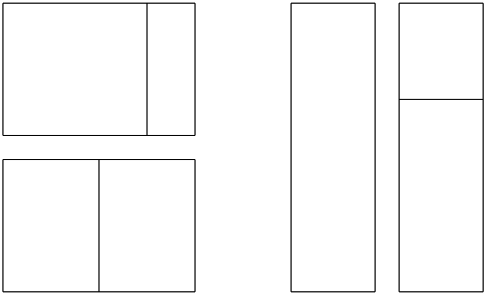

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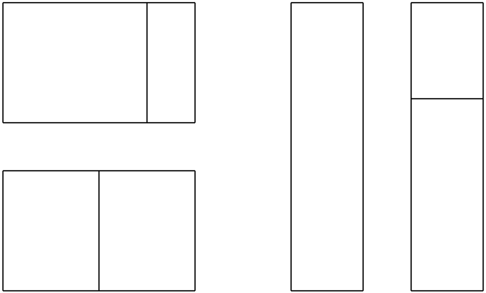

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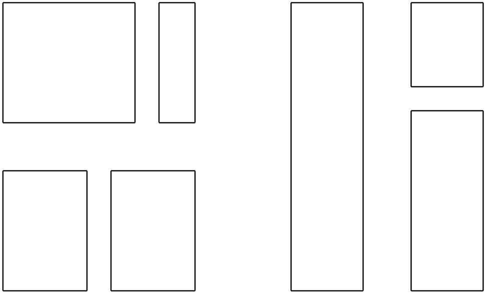

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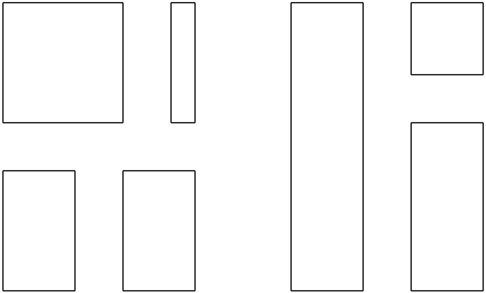

Characterization of guillotine partitions

A floorplane is guillotine if and only if it does not contain a pin-wheel structure.

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E. Ackerman, G. Barequet, R. Pinter, D. Romik:

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- Guillotine partitions $\longleftrightarrow$ Separable permutations

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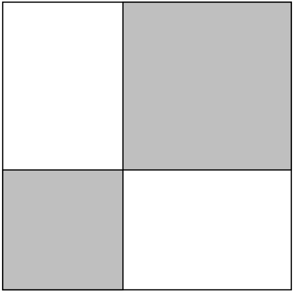

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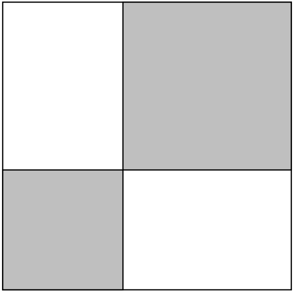

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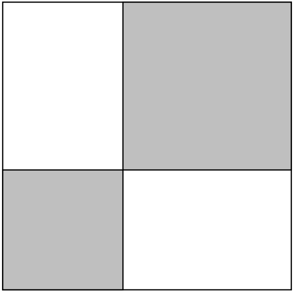

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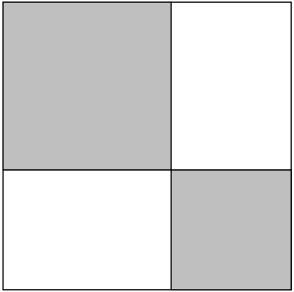

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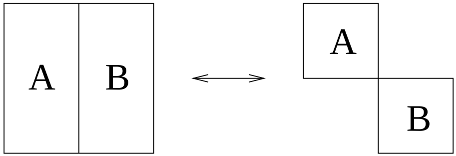
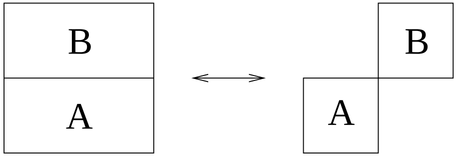
A bijection between Guillotine partitions with n cuts and Separable permutations of $[n+1]$





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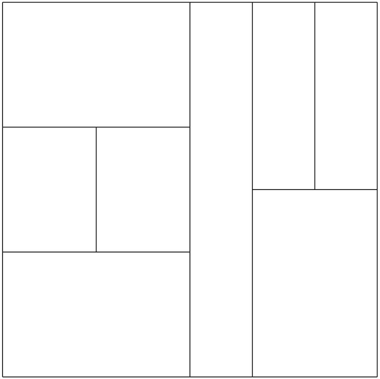

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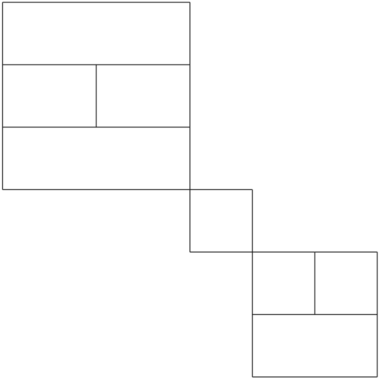

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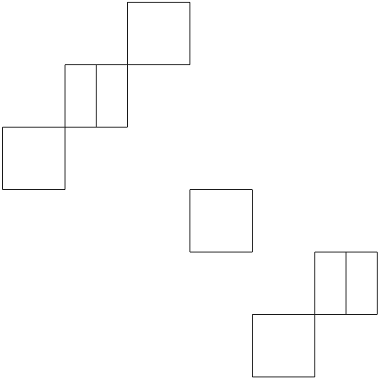

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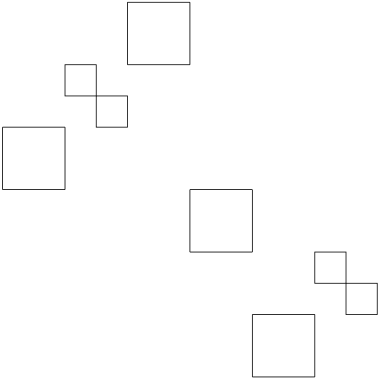

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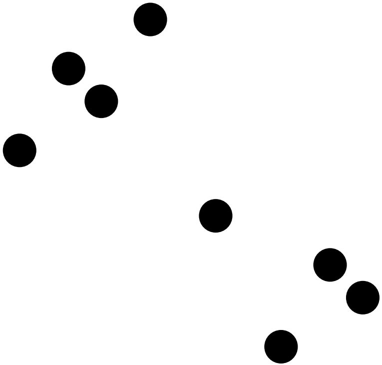

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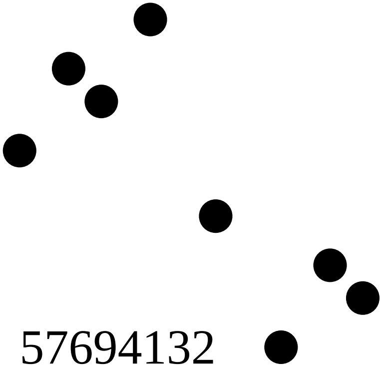

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- Usual permutations are 2-permutations (another approach exists).
- Geometrically, a subset of $[n]^d$. For each $1 \leq i \leq d$, $1 \leq j \leq n$, the hyperplane $x_i = j$ contains exactly one point from the set.

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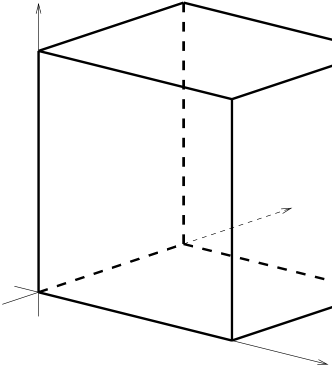
Separable d-permutations (d=3)



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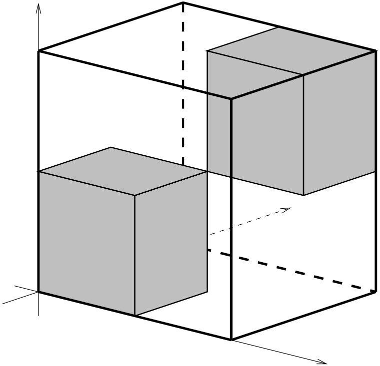

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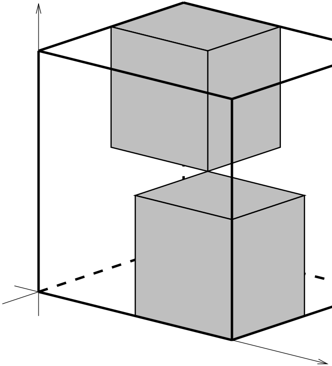

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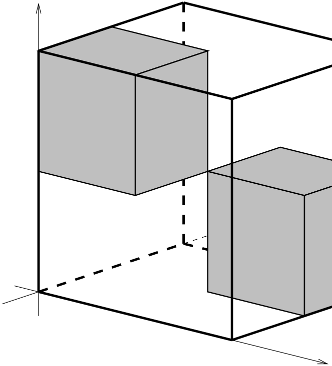

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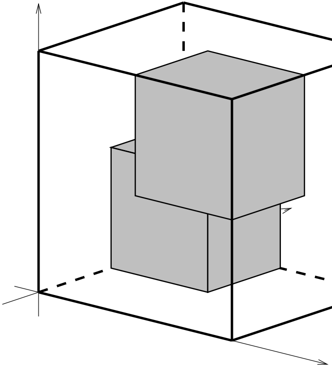

Generating function for the number of separable d-permutations:
$f = 1 + 2^{d-1} x f \cdot \left(1 + \frac{2^{d-1}-1}{2^{d-1}}(f-1)\right)$

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With 2^{d-1} replaced by d, this is the generating function for the number of guillotine partitions of a d-box.

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A bijection between
guillotine partitions of a 2^{d-1}-box with n cuts
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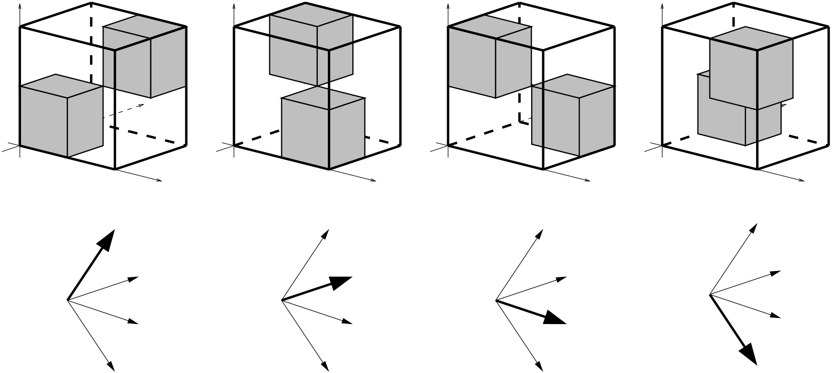
Consider a correspondence between axes of $\mathbb{R}^{2^{d-1}}$ and pairs of opposite orthants in $\mathbb{R}^d$.



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Given a guillotine partition of a 2^{d-1}-box,
Consider the "last" cut,
Suppose it is orthogonal to the x_j axis,
Take d-permutations corresponding to the lower and the upper parts of the partition,
and concatenate them using the pair of orthants which corresponds to x_j.

Characterization of separable d-permutations by forbidden patterns

Clearly, $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ are forbidden.

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Who else?

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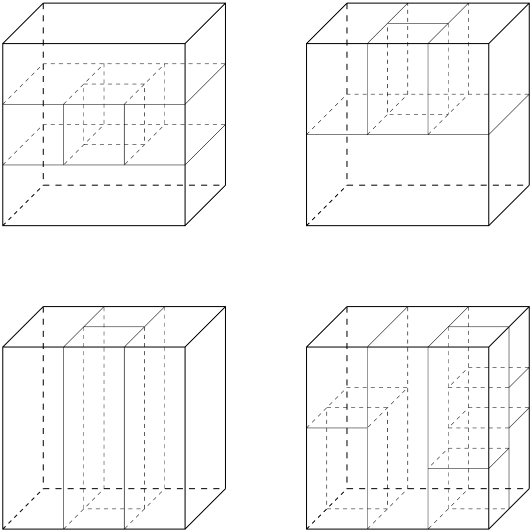
That's all!

Boundary guillotine partitions

Consider a guillotine partition of a d-dimensional cube.
Each cut is a $(d-1)$-dimensional cube.
It has $d-1$ pairs of $(d-2)$-dimensional faces.
If at least one member of each such pair is on the boundary of the cube,
the partition is a *boundary* partition.

Boundary guillotine partitions



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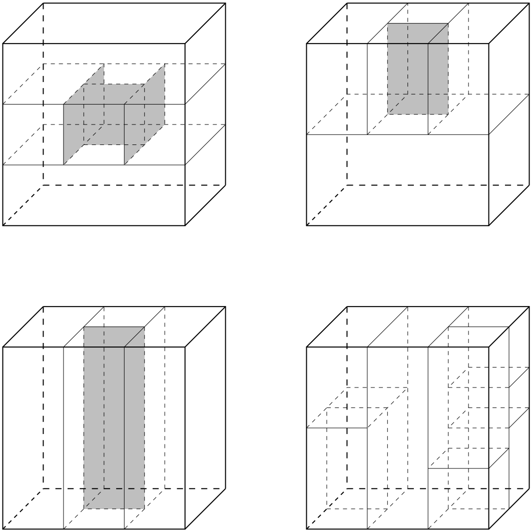

Enumeration of boundary guillotine partitions

Generating function:

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For $d = 2$:

$$f = 1 + \frac{2x(1-x)}{(1-x)^2}\left(1 + \frac{x(1-x)}{(1-2x)^2}\right)^2.$$

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Generating function:
For $d = 2$:

$$f = 1 + \frac{2x(1-x)}{(1-x)^2}\left(1 + \frac{x(1-x)}{(1-2x)^2}\right)^2.$$

For $d = 3$:

$$f = 1 + \frac{3x(1-x)}{(1-x)^2}\left(1 + \frac{2x(1-x)}{(1-2x)^2}\left(1 + \frac{x(1-x)}{(1-3x)^2}\right)^2\right)^2.$$

Enumeration of boundary guillotine partitions

Generating function:
For $d = 2$:

$$f = 1 + \frac{2x(1-x)}{(1-x)^2}\left(1 + \frac{x(1-x)}{(1-2x)^2}\right)^2.$$

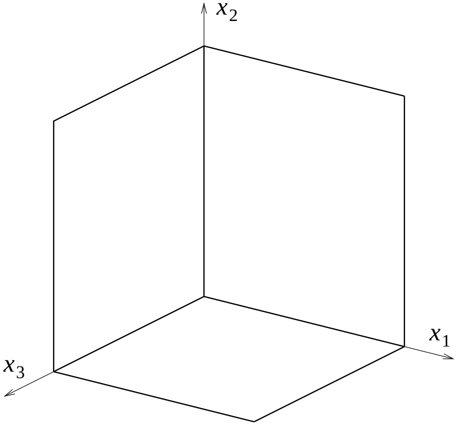
For $d = 3$:

$$f = 1 + \frac{3x(1-x)}{(1-x)^2}\left(1 + \frac{2x(1-x)}{(1-2x)^2}\left(1 + \frac{x(1-x)}{(1-3x)^2}\right)^2\right)^2.$$

For general d:

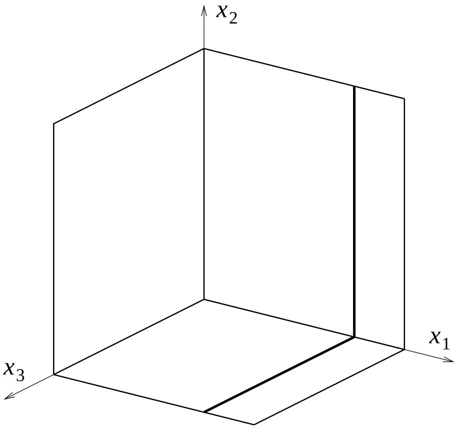
$$f = 1 + \frac{dx(1-x)}{(1-x)^2}\left(1 + \frac{(d-1)x(1-x)}{(1-2x)^2}\left(\dots\left(1 + \frac{2x(1-x)}{(1-(d-1)x)^2}\left(1 + \frac{x(1-x)}{(1-dx)^2}\right)^2\right)^2\dots\right)^2\right)^2.$$

Proof for $d = 3$.



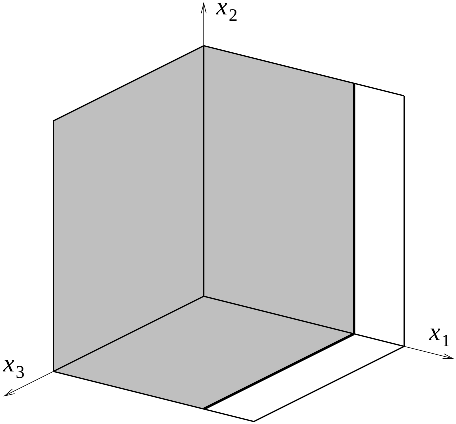
$f = 1 + \dots$

Proof for $d = 3$.



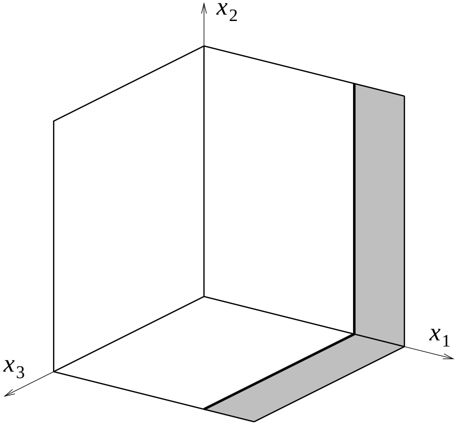
$f = 1 + 3x \ldots$

Proof for $d = 3$.



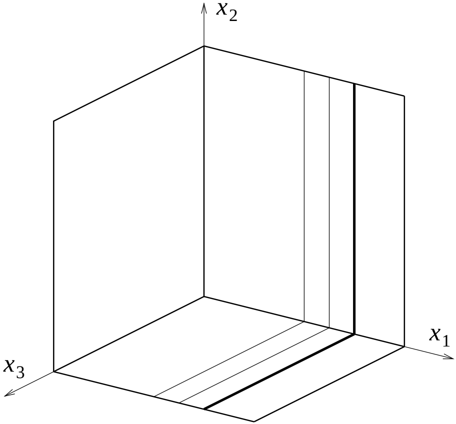
$f = 1 + 3xf^{-} \dots$

Proof for $d = 3$.



$f = 1 + 3xf^- f^+$

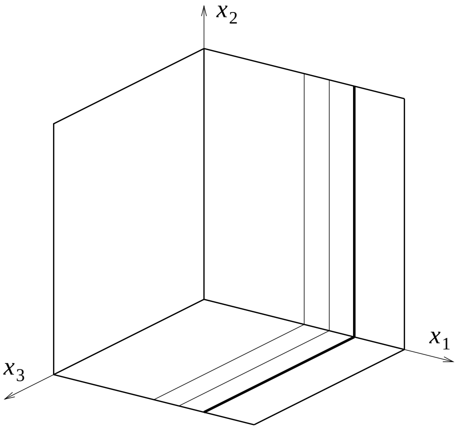
Proof for $d = 3$.



$$f = 1 + 3xf^{-}f^{+}$$

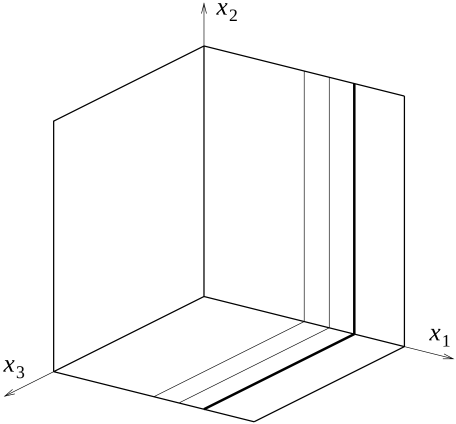
$$f^{-} = \frac{1}{1-x}f^{+}$$

Proof for $d = 3$.



$$f = 1 + 3x(1 - x)(f^{-})^2$$

Proof for $d = 3$.



$$f = 1 + 3x(1-x)(f^-)^2$$

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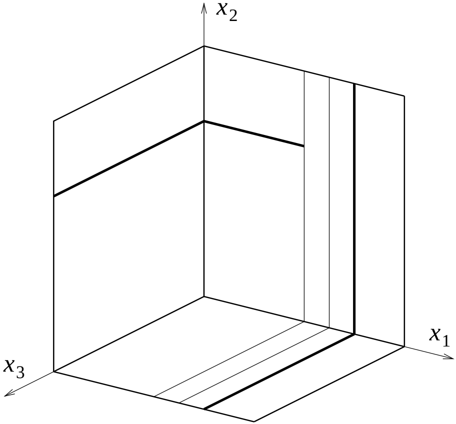
Proof for $d = 3$.



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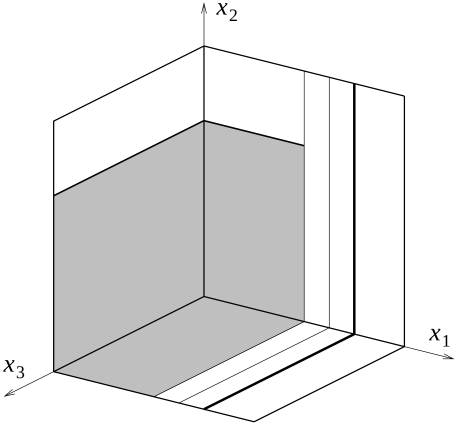

$$f = 1 + 3x(1-x)(f^-)^2$$

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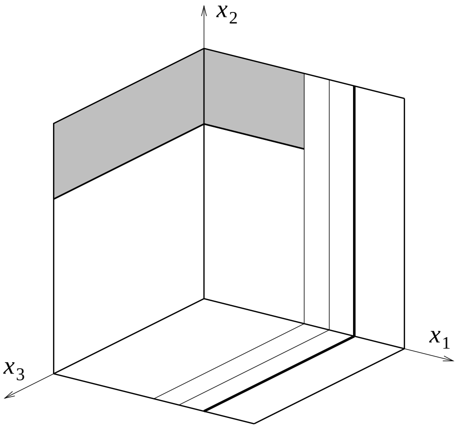

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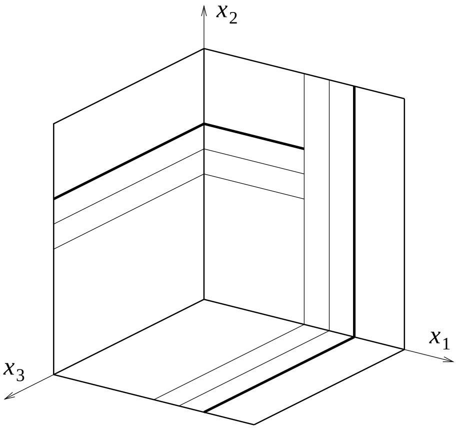

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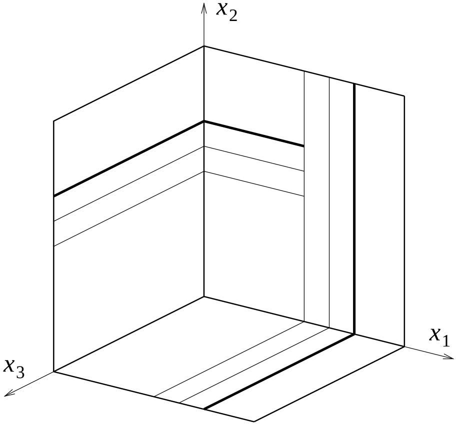

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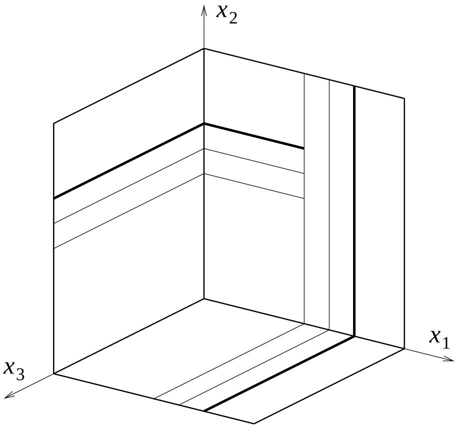

$$f = 1 + 3x(1-x)\left(\frac{1}{1-x}\left(1 + 2x(1-x)(f^{--})^2\right)\right)^2$$

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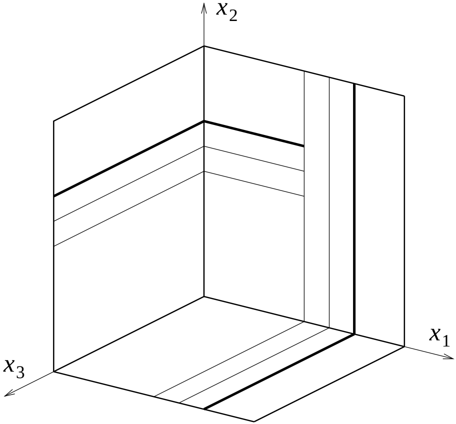

$$f = 1 + \frac{3x(1-x)}{(1-x)^2}\left(1 + 2x(1-x)(f^{--})^2\right)^2$$

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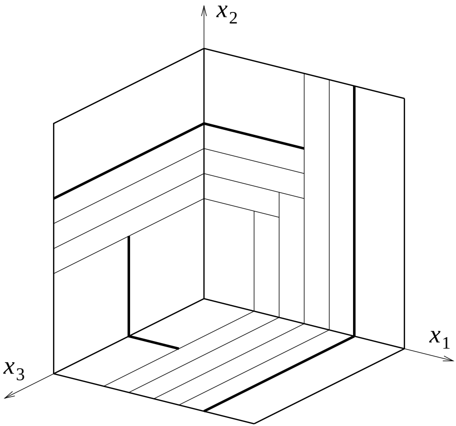

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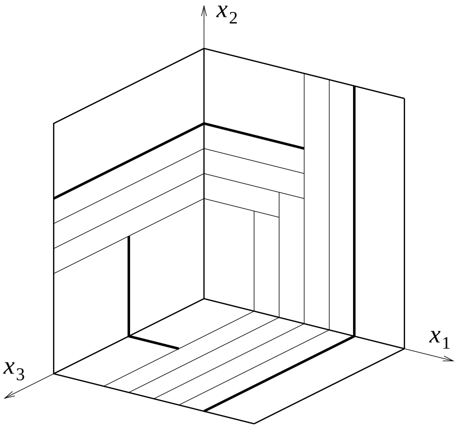

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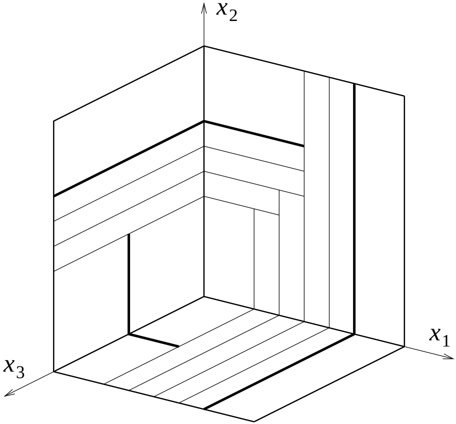

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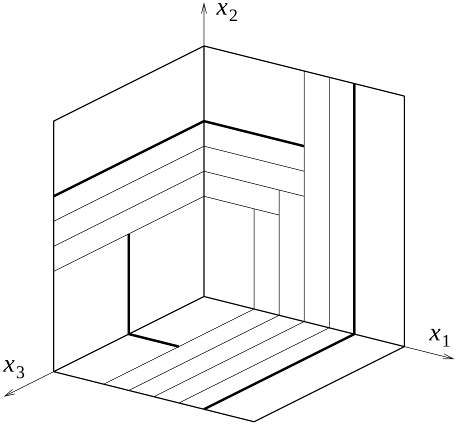

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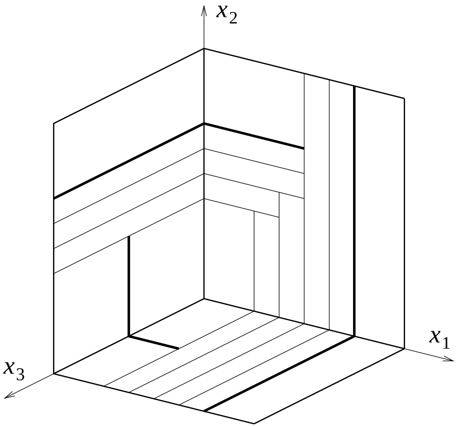

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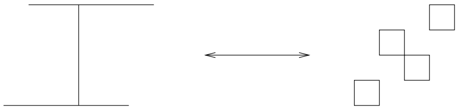
Planar boundary partitions in permutation patterns



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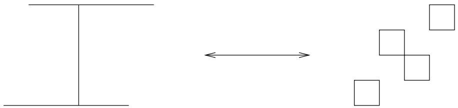
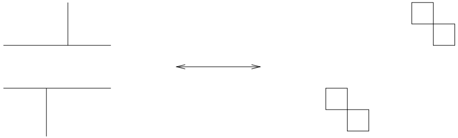


Permutations avoiding 2413, 3142, 1324, 4231 correspond to planar guillotine boundary partitions. The generating function of their enumerating sequence is

$$f = 1 + \frac{2x(1-3x+3x^2)^2}{(1-x)(1-2x)^4}.$$

The first ten terms are 1, 2, 6, 20, 64, 194, 562, 1570, 4258, 11266.

Planar boundary partitions in permutation patterns



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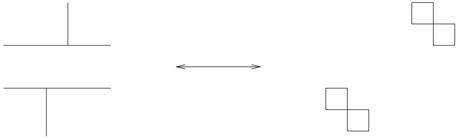

Permutations avoiding 2413, 3142, 2143, 3412 correspond to -avoiding planar guillotine partitions. The generating function of their enumerating sequence is

$$f = \frac{1-2x}{1-4x+2x^2}.$$

The first ten terms 1, 2, 6, 20, 68, 232, 792, 2704, 9232, 31520. This sequence is OEIS A006012.

A connection to threshold graphs (a remark by M. Golumbic)

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$(P_4, 2P_2, C_4)$-free graphs are threshold graphs.

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$(P_4, 2P_2, C_4)$-free graphs are threshold graphs.
Therefore: This sequence counts those permutation graphs which happen to be threshold graphs.

Questions for future research

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- Avoidance problems for multi-permutations.