Reduced Decompositions with Few Repetitions and Permutation Patterns

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June 16-20, 2008
Permutation Patterns 2008
University of Otago, Dunedin, New Zealand

Outline

- Definitions
- Motivation
- Reduced Decompositions with One Repetition
- Reduced Decompositions with Two Repetitions
- Current Questions

Definition ((Strong) Bruhat Order)

Define a partial order $\leq_B$ on the symmetric group S_n as follows: given π, $\sigma \in S_n$, we say $\pi \leq_B \sigma$ if and only if there exists a sequence of moves each interchanging the two elements of an inversion that transforms σ into π. This order is called the (Strong) Bruhat Order.

Example

$213465 \leq_B 312564$ in S_6

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$315264 \to 312564$

Example

$213465 \leq_B 312564$ in S_6

$315264 \longrightarrow 312564$

Example

$213465 \leq_B 312564$ in S_6

$315264 \rightarrow 312564$

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$213465 \leq_B 312564$ in S_6

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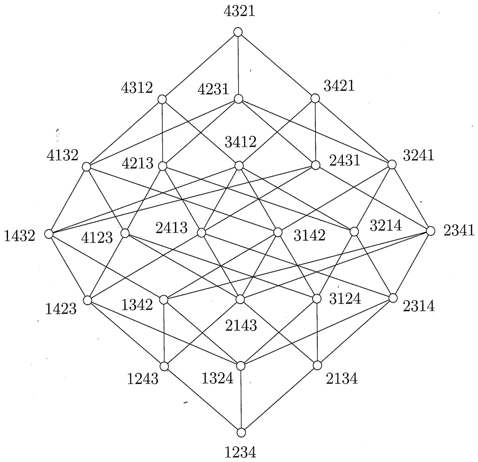
Bruhat Order of S_4



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Definition

A *reduced decomposition of* $\pi \in S_n$ is a sequence of transpositions $t_1 t_2 \ldots t_k$ each of the form $(i, i+1)$ for some i, such that $\pi = t_1 \ldots t_k$ and k is minimal with respect to this property.

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Example

(23)(12)(23) is a reduced decomposition for 321, but (23)(12)(23)(23)(23) is not.

Remark: Reduced decompositions are not unique! (12)(23)(12) and (23)(12)(23) are both reduced decompositions for 321.

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Definition

The number of elements in a reduced decomposition of $\pi \in S_n$ is called the *length* of π and written $l(\pi)$.

Example

$321 = (12)(23)(12)$ and $l(321) = 3$.

Notation: For purposes of this talk, rewrite reduced decompositions using only the first element of each transposition and I will put brackets around these expressions to distinguish them from permutations.

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$[23612] = (23)(34)(67)(12)(23)$

Given any reduced decomposition $[t_1 \dots t_l]$ for π, one can obtain a new reduced decomposition for π using braid moves.

Braid Moves:

- $[ij] = [ji]$ if $|i - j| > 1$
- $[i(i+1)i] = [(i+1)i(i+1)]$ $\forall i$

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Theorem

Let $[j_1 \ldots j_l]$ and $[k_1 \ldots k_l]$ be reduced decompositions for π. Then $[j_1 \ldots j_l]$ can be transformed into $[k_1 \ldots k_l]$ using braid moves.

Definition

Let $[t_1 \ldots t_l]$ be a reduced decomposition for π and let $1 \leq i_1 < i_2 < \ldots i_k \leq l$, then $t_{i_1} \ldots t_{i_k}$ is called a *subword* of $[t_1 \ldots t_l]$.

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Definition

A *factor* in a reduced decomposition is a subword whose elements are consecutive.

Example

Consider the reduced decomposition [12345]. [135] and [123] are subwords, but only [123] is a factor.

Theorem (Subword Property)

Let π, $\sigma \in S_n$ and let $[t_1 \dots t_k]$ be a reduced decomposition for σ. Then $\pi \leq_B \sigma$ if and only if there exists subword $[t_{i_1} t_{i_2} \dots t_{i_l}]$ of $[t_1 \dots t_k]$ that is a reduced decomposition for π.

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$213465 \leq 315264$ since:

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Can pattern avoidance help describe the structure of the Bruhat Order on the Symmetric Group?

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Theorem (Tenner 2007)

Let $\pi \in S_n$. The downset of π in the Bruhat order is a Boolean Algebra if and only if π avoids 3412 and 321.

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- Consider the reduced decompositions of 321 and 3412
- $\mathcal{R}(321) = \{[121], [212]\}$
- $\mathcal{R}(3412) = \{[2132], [2312]\}$
- Now consider the reduced decomposition of a permutation avoiding 321 and 3412.
- Consider $\mathcal{R}(24153) = \{[1324], [3124], [3142], [1342], [3412]\}$.

Why is this theorem true?

- Consider the reduced decompositions of 321 and 3412
- $\mathcal{R}(321) = \{[121], [212]\}$
- $\mathcal{R}(3412) = \{[2132], [2312]\}$
- Now consider the reduced decomposition of a permutation avoiding 321 and 3412.
- Consider $\mathcal{R}(24153) = \{[1324], [3124], [3142], [1342], [3412]\}$.
- There is one repeated element in the reduced decompositions of 321 and 3412 and there are no repeated elements in the permutation that avoids 321 and 3412

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π avoids 321 and 3412 if and only if there exists a reduced decomposition of π with no repeated elements.

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Remark: This means every reduced decomposition of π has no repeated elements.
If a reduced decomposition of π has no repeated elements, then every subword will be a reduced decomposition and no two subwords will produce the same permutation.

Example

Subwords of $[123] : \{[13], [12], [23], [1], [2], [3], \emptyset\}$.

Subwords of $[121] : \{[12], [21], 11, [1], [2], \emptyset\}$

Question: What can we say about permutations with a reduced decomposition with exactly one repetition?

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Theorem

$\pi \in S_n$ has a reduced decomposition with exactly one element repeated if and only if π avoids 3412 and contains exactly one 321 pattern or π avoids 321 and contains exactly one 3412 pattern.

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Theorem

$\pi \in S_n$ has a reduced decomposition with exactly one element repeated if and only if π avoids 3412 and contains exactly one 321 pattern or π avoids 321 and contains exactly one 3412 pattern. More specifically,

- *π contains exactly one 321 pattern and avoids 3412 if and only if π has a reduced decomposition with $[i(i+1)i]$ as a factor for some $i \in \{1, \ldots n-2\}$ and no other repetitions.*

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- *π contains exactly one 321 pattern and avoids 3412 if and only if π has a reduced decomposition with $[i(i+1)i]$ as a factor for some $i \in \{1, \ldots n-2\}$ and no other repetitions.*
- *π contains exactly one 3412 pattern and avoids 321 if and only if π has a reduced decomposition with $[i(i-1)(i+1)i]$ as a factor for some $i \in \{2, \ldots n-2\}$ and no other repetitions.*

Examples

- 25314 has r.d. [41232]
  ([232] is of the form $[i(i+1)i]$, so there must be a 321.)

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  ([232] is of the form $[i(i+1)i]$, so there must be a 321.)
- 34152 has r.d. [21324]
  ([2132] is of the form $[i(i-1)(i+1)i]$, so there must be a 3412.)
- 214365 has r.d.s $\{[135],[153],[315],[351],[513],[531]\}$

Can we count the number of permutations whose reduced decompositions have exactly one repetition?

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By computer experiment we have:

n	# of $\pi \in S_n$ that avoid 3412 and contain exactly one 321	# of $\pi \in S_n$ that avoid 321 and contain exactly one 3412
3	1	0
4	6	1
5	25	6
6	90	25
7	300	90
8	954	300

Theorem (D.)

The number of permutations in S_n that avoid 3412 and contain exactly one 321 pattern is equal to the number of permutations in S_{n+1} that avoid 321 and contain exactly one 3412.

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$215463 \longrightarrow [14345]$

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Example:

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Example:

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How many permutations are there that have a reduced decomposition with exactly one factor of the form [i(i+1)i] ? Equivalently, how many permutations in S_n avoid 3412 and contain exactly one 321?

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Equivalently, how many permutations in S_n avoid 3412 and contain exactly one 321?
The first few such permutations in S_n have the following reduced decompositions (up to equivalence):

n	$\mathcal{R}$
3	[121]
4	[121], [3121], [1213], [232], [1232], [2321]
5	[121], [3121], [1213], [4121], [43121], [31214], [12134], [41213], [232], [1232], [2321], [4232], [2324], [41232], [12324], [42321], [23214], [343], [1343], [2343], [3432], [12343], [23431], [13432], [34321]

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  - For $i = 2$,
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    2. for each r.d. already built from step 1 create two new r.d.'s by putting a 1 at the beginning and a 1 at the end. This gives $\{[1232], [2321]\}$.

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  - For $i = 1$, the only such reduced decomposition is $[121]$.
  - For $i = 2$,
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    2. for each r.d. already built from step 1 create two new r.d.'s by putting a 1 at the beginning and a 1 at the end. This gives $\{[1232], [2321]\}$.
    3. Therefore, the reduced decompositions for $i = 2$ are: $\{[232], [1232], [2321]\}$.

Partial Sketch of Count:

- For $i = 1$, r.d.s: $\{[121]\}$.
- For $i = 2$, r.d.s: $\{[232], [1232], [2321]\}$.
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  1. for each r.d. built from $i = 2$, add one to each element. This gives $\{[343], [2343], [3432]\}$.
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  3. for each r.d. already built from step 1 and having a 2 in it, create two new r.d.s by putting a 1 at the beginning and a 1 at the end. This gives $\{[12343], [23431], [13432], [34321]\}$.
- If $a(i)$ is the number of such reduced decompositions for a given i, then $a(i) = a(i-1) + a(i-2) + 2(a(i-1) - a(i-2))$.

Partial Sketch of Count:

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- For a fixed i, the number of reduced decompositions with factor $[i(i+1)i]$ is $F_{2i}F_{2(n-i-1)}$.

Theorem (D.)

The number of permutations in S_n that avoid 3412 and contain exactly one 321 is

$$\sum_{i=1}^{n-2} F_{2i} F_{2(n-i-1)}$$

where F_m is the m^{th} Fibonacci number.

Closed Form:

$$\sum_{i=1}^{n-2} F_{2i}F_{2(n-i-1)} = \frac{2(2n-5)F_{2n-6} + (7n-16)F_{2n-5}}{5}$$

Closed Form:

$$\sum_{i=1}^{n-2} F_{2i}F_{2(n-i-1)} = \frac{2(2n-5)F_{2n-6} + (7n-16)F_{2n-5}}{5}$$

Generating Function:

$$\frac{x^3}{(1-3x+x^2)^2}$$

We can use reduced decompositions to count permutations that avoid 3412 and contain exactly one 321. Can we use them to count involutions of the same form?

Theorem (Egge 2003)

The number of involutions in S_n which avoid 3412 and contain exactly one copy of 321 is

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Remark: This theorem was proven using Chebyshev Polynomials

Theorem (D.)

Let $\pi \in S_n$. π is an involution avoiding 3412 and containing exactly one 321 if and only if π has a reduced decomposition $[t_1 \ldots t_k]$ that has a factor of the form $[i(i+1)i]$ for some $i \in \{1, \ldots, n-2\}$ and no other repetitions and if t_j is an element of the reduced decomposition other than i or $i+1$, then $|t_j - t| > 1$ for all t in the reduced decomposition.

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Example

[34317], [74542], [121468]

Counting the number of such reduced decompositions gives:

Theorem

The number of involutions in S_n that avoid 3412 and contain exactly one 321 is

$$\sum_{i=1}^{n-2} F_i F_{(n-i-1)}$$

where F_m is the m^{th} Fibonacci number.

Summary of Counting Results

Theorem

The number of permutations in S_n that avoid 3412 and contain exactly one 321 is

$$\sum_{i=1}^{n-2} F_{2i}F_{2(n-i-1)}$$

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Definition

A factor of a reduced decomposition with exactly two repetitions is *twisted* if it cannot be transformed into a factor of the form $[i - -i - -j - -j]$ using braid moves.

Twisted Factors of Length 5

- $[i(i-1)(i+1)i(i+1)]$ $[21323] = 3421$
- $[(i+1)i(i+1)(i-1)i]$ $[32312] = 4312$
- $[(i+1)i(i-1)i(i+1)]$ $[32123] = 4231$

Twisted Factors of Length > 5

Length 6

- $[i(i-1)(i+1)i(i+2)(i+1)]$ $[213243] = 34512$
- $[(i+1)(i+2)i(i+1)(i-1)i]$ $[342312] = 45123$
- $[i(i-1)(i+1)(i+2)(i+1)i]$ $[213432] = 35142$
- $[(i+1)(i+2)i(i-1)i(i+1)]$ $[342123] = 42513$

Length 7

- $[i(i-1)(i+2)(i+1)(i+3)(i+2)i]$ $[2143542] = 351624$

Theorem (D.)

A factor of a reduced decomposition with exactly two repeated elements has a subfactor that is equivalent to one of the previous twisted factors or has a subfactor that is of the form $[i - -i - -j - -j]$. *In particular, any twisted factor is equivalent to one that has been previously listed.*

Theorem (D.)

π has a reduced decomposition with a factor of the form

1. $[i(i-1)(i+1)i(i+1)]$
2. $[(i+1)i(i+1)(i-1)i]$
3. $[(i+1)i(i-1)i(i+1)]$

if and only if π has exactly two 321 patterns of the corresponding form

1. 3421
2. 4312
3. 4231

and avoids 3412.

This classifies all twisted factors of length 5.

Theorem (D.)

The following quantities are equal:

- $|\{\pi \in S_n :$ π *avoids 3412 and contains exactly two 321 patterns of the form 3421*$\}|$
- $|\{\pi \in S_n :$ π *avoids 3412 and contains exactly two 321 patterns of the form 4312*$\}|$
- $|\{\pi \in S_n :$ π *avoids 3412 and contains exactly two 321 patterns of the form 4231*$\}|$
- $\sum_{i=1}^{n-3} F_{2i}F_{2(n-i-1)}$ *where* F_m *is the* m^{th} *Fibonacci number*

Corollary

The number of permutations in $S_n(3412)$ that contain exactly two 321 patterns sharing two of the three elements is

$$3\sum_{i=1}^{n-3} F_{2i}F_{2(n-i-1)}$$

where F_m is the m^{th} Fibonacci number.

Current Questions

- What are the characterizations for the other twisted factors? Can we characterize all of the permutations in S_n that
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- Do these characterizations of reduced decompositions aid in characterizing Bruhat intervals?

Thank You!

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